MST121 RP

The Open University

USING MATHEMATICS

Revision Pack

REVISION PACK

SUP 69499 7
3.1

Contents

How to use this pack

This pack has been designed to help you to make the most efficient use of whatever time you have for revision. It is in two main parts: an audit section and a revision section. There is also an index and a bookmark (the back cover of this pack).

The *Guide to Preparation* contains detailed advice about your general preparation for the course(s) and for your mathematical preparation in particular. **You should work through the *Guide to Preparation*, and use this pack as directed there.**

Audit Section

This section contains the Diagnostic Quiz and its Solutions.

Revision Section

This section comprises revision notes, worked examples and practice exercises with solutions.

The index is there to help you find topics, but can also be a checklist of mathematical words which you should understand. If you come across words or symbols which are unfamiliar, then perhaps you should start your own mathematical dictionary.

The back cover can be used as a bookmark, and includes hints on making the most of your revision time and what to do when you are 'stuck'.

Diagnostic Quiz

The questions that follow are designed to give you a look at a number of mathematical techniques that will be required in MST121 (and MS221).

The *Guide to Preparation* explains how to make best use of the *Revision Pack* and in particular this quiz.

Don't worry. This quiz is not a test or an examination. Only you will know how well or how badly you did.

Don't hurry. You can take as long as you like, and do the questions in any order. You may well find that in any one question you can do one part and then get stuck or slip up on another.

As you work out your answers, keep a note about how confident you feel about them. When you have finished the questions, check your answers with the solutions, which start on page 13. The solutions contain references to the appropriate sections of the Revision section, so you can then spend some time working on the topics with which you had difficulty.

You may find it helpful to return to some questions after you have revised a topic to check that you are then able to answer them correctly. The questions cannot cover everything included in the Revision section, so even if you get them all right, you may still find it helpful to read through that section.

Good luck – enjoy the challenge!

1 Numbers

Question 1

The use of powers is very common in mathematics, and it is important to know what they mean. See how many of the following you can do. If you get stuck on one, try the next one.

(a) Evaluate each of the following, without using your calculator.

$$3^2,\quad 8^3,\quad 4^{-1},\quad (\tfrac{1}{2})^2,\quad (-8)^2,\quad (-2)^3,\quad 3^{-2},\quad (-3)^2,\quad -3^2.$$

(b) Use your calculator to find each of the following.

(i) 7^6 (ii) 3.2^4 (iii) $(-3.2)^4$ (iv) $-(3.2)^4$

(v) $(-3.2)^{-4}$ correct to two significant figures

Question 2

(a) After carrying out a series of numerical operations, a calculator gives the following answer.

1.917150743E−5

Explain what this means.

(b) How would you expect the same calculator to display the following number?

32 190 818 670

Question 3

It is very easy to make a mistake when you use a calculator, so it is important to check that your answer is reasonable. You can do this by estimating the answer.

(a) Give rough estimates for each of the following numbers.

(i) 413 (ii) 2782 (iii) 12.4 (iv) 0.1253 (v) 189 025

(b) Use your estimates to obtain an approximate answer to each of the calculations below.

(i) 413×2782 (ii) $413 \times 189\,025$ (iii) $\dfrac{12.4}{0.1253}$

(iv) $\dfrac{2782 \times 0.1253}{12.4}$

Check the accuracy of your answers by using a calculator.

Question 4

Knowing the correct order in which to deal with the arithmetic operations in an expression is important.

Calculate each of the following.

(a) $32 + 2 \times 5 + 6$ (b) $(32 + 2) \times 5 + 6$ (c) $32 + 2(5 + 6)$

(d) $(32 + 2) \times 5^2 + 6$ (e) $(32 + 2)(5^2 + 6)$

Question 5

By finding the prime factors of each of the numbers 12, 20 and 45, find the smallest whole number that can be divided exactly by all three numbers.

Question 6

Sometimes numbers look very different, but are actually the same.

(a) Use a calculator to write each of the following fractions as decimals, correct to three decimal places.

(i) $\dfrac{3}{40}$ (ii) $\dfrac{15}{62}$ (iii) $1\dfrac{13}{84}$ (iv) $21\dfrac{87}{93}$

(b) Without using your calculator, evaluate each of the following.

(i) $\dfrac{2}{3} + \dfrac{2}{7}$ (ii) $\dfrac{5}{6} - \dfrac{2}{3}$ (iii) $\dfrac{4}{15} + \dfrac{7}{20}$ (iv) $3\dfrac{2}{5} - 2\dfrac{7}{8}$

(v) $\dfrac{3}{8} \times 1\dfrac{5}{11}$ (vi) $1\dfrac{3}{4} \div 4\dfrac{2}{3}$

Question 7

In this question, try to use your calculator only when really necessary, and then with as few key strokes as possible.

Find the reciprocal of each of the following numbers, giving answers that are not exact correct to three significant figures.

(a) 10 (b) 5 (c) 10^{-1} (d) 72.5 (e) 0.003 56

Question 8

Percentages occur in many walks of life, including MST121!

(a) Increase each of the following numbers by the percentage shown.

(i) 14 by 10% (ii) 142 by 115%

(b) By what percentage has £300 been increased or decreased to arrive at the following answers?

(i) £317.5 (ii) £60

Question 9

(a) Evaluate each of the following without using your calculator. Then see if you can obtain the same answer by using your calculator.

(i) $3 + (-4) - 6$ (ii) $3 - 4 + (-6)$ (iii) $3 + (-4 + 6)$

(iv) $3 \times (-4 + 6)$ (v) $(-6) \div (-5)$ (vi) $6 \div (3 \div 5)$

(b) Calculate each of the following, without using your calculator.

(i) $\sqrt{14\,400}$ (ii) $\sqrt{0.81}$ (iii) $\sqrt{12}$ (iv) $\sqrt[3]{24}$

(v) $\sqrt{\dfrac{1}{10\,000}}$

Question 10

Without using your calculator, write each of the following numbers in $\log_{10}$ form. For example, $\log_{10} 100 = 2$.

(a) 10^{-2} (b) $\dfrac{1}{100}$ (c) 0.001

Question 11

(a) Use your calculator to find each of the following, giving your answers correct to four decimal places.

(i) $\ln 3.142$ (ii) $\ln 14.16$ (iii) $\ln 14\,658$ (iv) $\ln 0.0324$

(b) Use your calculator to find the numbers, correct to three significant figures, whose natural logarithms are as follows.

(i) 0.812 (ii) 1.623 (iii) 2.976 (iv) 0.0356

Question 12

Three people decide to split a food bill between them, taking into account the number of meals eaten at home. They agree that it should be split in the ratio 2 to 3 to 5. The bill is £70.

How much does each person pay?

2 Measures

Question 13

(a) Mark the following numbers on a number line, and hence arrange them in ascending order.

2.9, 4.3, −0.2, −3, −3.5, 1.5.

(b) In each of the following parts, insert the appropriate $>$ or $<$ sign between the numbers to make a true statement. Your number line for part (a) may help you.

(i) 2.9 4.3 (ii) −0.2 1.5 (iii) 2.9 1.5 (iv) −0.2 −3

Question 14

Make x the subject of the formula in each of the following equations.

(a) $2x = y - 3$ (b) $p = x^2 + 3$ (c) $(x + 3)^2 = t$

(d) $y = \frac{1}{2}\sqrt{x}$ (e) $m = \frac{3p}{x^2}$

Question 15

In mathematics we often use radians, rather than degrees, to measure angles.

(a) How many radians are there in one complete turn?

(b) Sketch an angle of 60°. What is the measure of this angle in radians?

(c) Sketch an angle of $\frac{3\pi}{2}$ radians. What is the measure of this angle in degrees?

Question 16

The word 'average' has more than one meaning. Some matchboxes are labelled 'Average contents 50 matches'.

(a) The contents of five boxes of matches were checked and found to be:

50, 47, 48, 51, 51.

Find the mean, median and mode of the contents of the five boxes.

(b) The contents of a further five boxes were checked and found to be:

50, 48, 48, 51, 48.

Combine these figures with those from part (a), and calculate the mean, median and mode for the ten boxes.

3 Some basic figures

Question 17

It is important to be able to obtain answers from information in a diagram.

In the rectangle in Figure 0.1, the side AB is of length 6 m and the side BC is of length 8 m. You may find it helpful to insert these figures in the diagram.

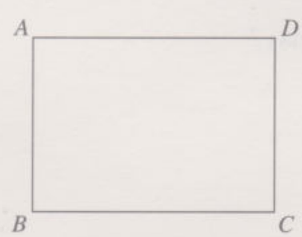


Figure 0.1

(a) Use Pythagoras' Theorem to find the length of side AC.

(b) Calculate the perimeter of the triangle ABC.

(c) Find the area of the rectangle, and hence the area of the triangle ABC.

(d) Confirm that the area of the triangle you have obtained in part (c) is correct, by finding the area directly.

(e) A path of width one metre is constructed whose inner edge is the rectangle $ABCD$ and whose outer edge is a larger rectangle. What is the area of the path?

Question 18

In Figure 0.2, O is the centre of a circle, and A, B, C and D lie on that circle.

The line segments AC and AB are not equal in length, and AOD is a diameter.

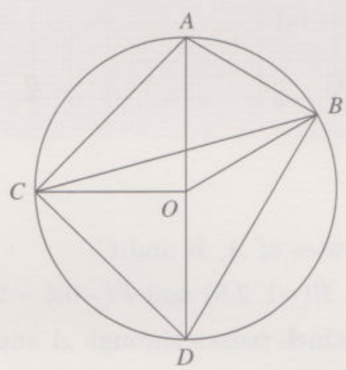


Figure 0.2

(a) Explain why angle ACD is 90°.

(b) Identify

(i) three isosceles triangles (there are more than three);

(ii) two scalene triangles (there are more than two);

(iii) two right-angled triangles.

(c) If angle DAC is 40° and angle CBO is 15°, show that angle COD is 80° and find the other angles in the figure.

(d) If the radius of the circle is 10 cm, find, correct to the nearest whole number:

(i) the area of the triangle OCD;

(ii) the length of the arc CD.

(e) What is the area of the sector OCD to the nearest whole number?

4 Coordinates and lines

Question 19

Coordinates are a shorthand way of describing the positions of points relative to a fixed point (the origin) and a pair of axes. The points A, B and C are plotted on Figure 0.3.

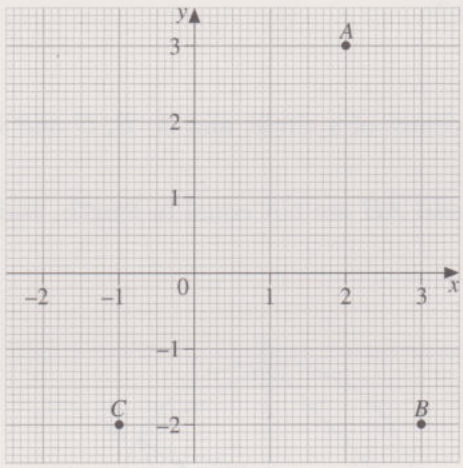


Figure 0.3

(a) Write down the coordinates of A, B and C.

(b) Plot the points $D(1, 3)$, $E(-1, 2.5)$ and $F(-0.5, -2)$.

(c) Draw the straight line which passes through A and F.

(i) What is the gradient of this line?

(ii) Where does it cross the x-axis?

(iii) Where does it cross the y-axis?

5 Algebra

Question 20

If $x = -8$, find the value of each of the following expressions.

(a) (i) $5x$ (ii) $-5x$ (iii) $3x - 2$ (iv) $3(x + 2)$

(b) (i) $x^2 + 4$ (ii) $(x + 4)^2$ (iii) $(x + 24)^{\frac{1}{2}}$ (iv) $x^2 - 3x - 4$

Question 21

Simplify each of the following expressions as far as possible.

(a) $2a + 3b - 4a$ (b) $2a + 3b - 4c - (a - 5b)$

(c) $2a + 3b - 4c + 2(a - 5b)$ (d) $2a + 3b - 4c - 2(a - 5b)$

(e) $2a + 3b - 4c + c(2 - 5b)$

Question 22

Multiply out the brackets in each of the following expressions, and simplify the result as far as possible.

(a) $(b + 1)(b + 2)$ (b) $(c - 2)(c + 5)$ (c) $(d - f)^2$

(d) $(3x + 4)(2x - 7)$ (e) $(3x - 4y)(5y + 6x)$

Question 23

Factorising is the opposite of multiplying out. You start with the sum of a number of terms and aim to express it as a product. Factorise each of the following expressions.

(a) $ab + a$ (b) $ab + ac$ (c) $ab + 2ac + a^2$ (d) $a^2 - b^2$
(e) $a^2 + 2ab + b^2$ (f) $6a^2 + a - 1$

Question 24

Solve each of the following equations.

(a) (i) $2x - 5 = 15$ (ii) $2(x - 5) = 15$ (iii) $2(x - 5)^2 = 32$
(iv) $8 - 2x = x + 7$

(b) (i) $p^2 + 2p - 4 = 0$ (ii) $36 = \dfrac{25}{v^2}$ (iii) $5^t = 26$

In parts (i) and (iii), give your answers correct to two decimal places.

Question 25

Solve the following pair of simultaneous equations.

$$x + 2y = 4$$
$$2x - 3y = -6$$

Question 26

Solve each of the following inequalities, and illustrate each answer on a number line.

(a) $2x - 3 \geq 5$ (b) $2x - 4 < 1$ (c) $3 - x > 5$

6 Trigonometry

Question 27

In the triangle in Figure 0.4, angle A is $90°$, angle B is $70°$ and the hypotenuse has length 15 cm.

Using trigonometric ratios, or otherwise, find the unknown sides and angle of the triangle. Give lengths correct to three significant figures.

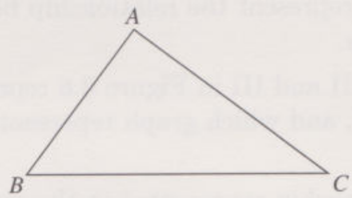


Figure 0.4

7 Graphs and functions

Question 28

The graph on the left in Figure 0.5 shows a journey in which Chris walks from home to the newsagent, buys a newspaper, and walks back, stopping to talk to a friend on the way. The speed of walking is constant in each section of the journey. Use the graph to deduce:

(a) (i) the distance, in kilometres, from Chris's home to the newsagent;

(ii) the speed of walking in each section of the journey, in kilometres per hour;

(iii) the distance, in kilometres, from the newsagent to the point at which the conversation with the friend takes place.

(b) Explain why the graph on the right of Figure 0.5 cannot represent a journey.

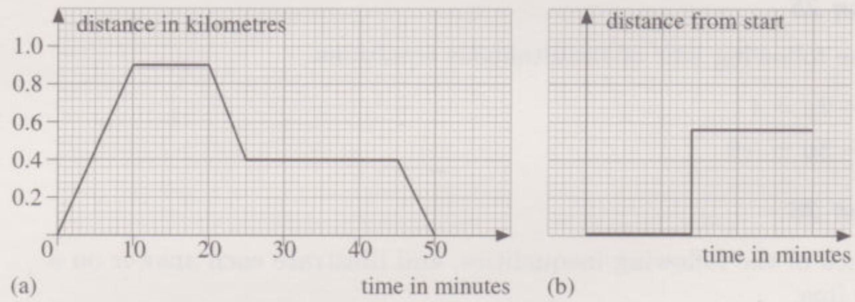


Figure 0.5

Question 29

In a building development all plots are rectangular. In one section (A) all plots will have an area of $500\,\mathrm{m}^2$. In another section (B) the areas of plots will vary, but the dimensions of each plot will be such that the length is $\frac{8}{5}$ of the breadth.

Let l metres and b metres represent the length and breadth, respectively, of a plot.

(a) For plots in section A, describe in words the relationship between the length and breadth, and represent this relationship in symbols.

(b) For plots in section B, represent the relationship between the length and breadth in symbols.

(c) (i) Which of graphs I, II and III in Figure 0.6 represents the relationship in part (a), and which graph represents the relationship in part (b)?

(ii) Describe the relationship represented in the graph you did not choose.

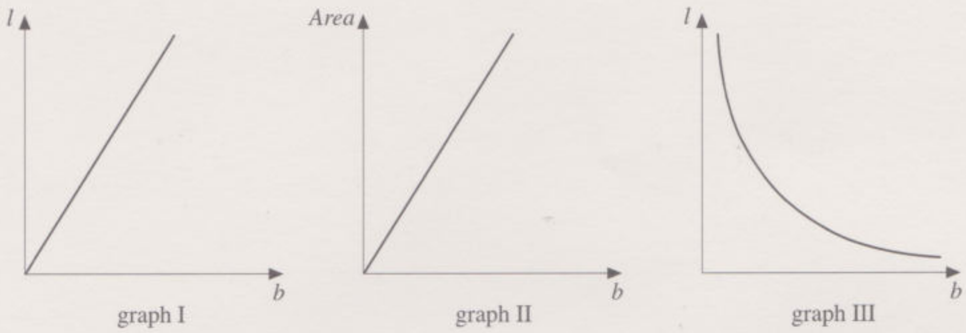


Figure 0.6

Question 30

Sketch the graph of each of the following equations. (Accurate plots are not required. No calculators allowed!)

(a) $y = x^2$ (b) $y = x^2 - 2$ (c) $y = (x-2)^2$ (d) $y = x^3$

Question 31

Sketch the graph of each of the following functions.

(a) $f(x) = \sin x$ for x between -2π and 2π

(b) $g(x) = 1 + \cos x$ for x between -2π and 2π

(c) $h(x) = e^x$

8 Geometry

Question 32

Two rectangular pieces of paper each measure 31 cm by 20 cm. One is used to make an open-ended tube of circular cross-section and length 20 cm. The other is used to form an open-ended prism, also of length 20 cm, whose cross-section is an equilateral triangle.

For each shape, 1 cm (of the 31 cm) is taken up by the need to secure the edges.

Figure 0.7

(a) Find:

(i) the radius of the circular cross-section and hence the area of the circular hole at the base of the cylinder;

(ii) the area of the triangular hole at the end of the prism;

(iii) the surface area of each shape;

(iv) the ratio of the volume of the cylinder to that of the prism formed by closing off the ends.

(b) Can you think of any everyday examples of such cylinders and prisms?

Solutions to Diagnostic Quiz

Solution 1

(a) $3^2 = 9$; $8^3 = 512$; $4^{-1} = \frac{1}{4}$; $(\frac{1}{2})^2 = \frac{1}{4}$; $(-8)^2 = 64$; $(-2)^3 = -8$; $3^{-2} = \frac{1}{9}$; $(-3)^2 = 9$; $-3^2 = -9$. See Section 1.1.

(b) (i) 117 649 (ii) 104.857 6 (iii) 104.857 6

(iv) −104.857 6 (v) 0.009 5 (to 2 s.f.)

Solution 2

This is the calculator's way of showing very large and very small numbers. See Section 1.1.

(a) 1.917150743E−5 means $1.917\,150\,743 \times 10^{-5}$, which is equal to 0.000 019 171 507 43.

(b) $32\,190\,818\,670 = 3.219\,081\,867\,0 \times 10^{10}$, which the calculator will show as 3.219081867E10.

Solution 3

You may have rounded the figures to other values, so your estimated answers could differ slightly from ours. The important thing is to make sure that you can estimate an answer in your head. See Section 1.2.

(a) (i) 400 (ii) 3000 (iii) 10 (iv) 1/10 (0.1) (v) 200 000

(b) (i) $400 \times 3000 = 1\,200\,000$;
calculated value $= 1\,148\,966$.

(ii) $400 \times 200\,000 = 80\,000\,000$;
calculated value $= 78\,067\,325$.

(iii) $10 \div \dfrac{1}{10} = 100$;
calculated value $= 98.962\,5$ (to 4 d.p.).

(iv) $\left(3000 \times \dfrac{1}{10}\right) \div 10 = 30$;
calculated value $= 28.111\,7$ (to 4 d.p.).

Solution 4

(a) $32 + 10 + 6 = 48$ (b) $34 \times 5 + 6 = 170 + 6 = 176$ See Section 1.2.

(c) $32 + 22 = 54$ (d) $34 \times 25 + 6 = 850 + 6 = 856$

(e) $34 \times (25 + 6) = 34 \times 31 = 1054$

Solution 5

$12 = 2 \times 2 \times 3$, $20 = 2 \times 2 \times 5$, $45 = 3 \times 3 \times 5$; so the smallest number divisible by 12, 20 and 45 is $2 \times 2 \times 3 \times 3 \times 5 = 180$. See Section 1.3.

Solution 6

(a) Each of the following answers is given correct to three decimal places.

See Section 1.4.

(i) $\frac{3}{40} = 0.075$ (ii) $\frac{15}{62} = 0.242$

(iii) $\frac{13}{84} = 0.155$, so $1\frac{13}{84} = 1.155$.

(iv) $\frac{87}{93} = 0.935$, so $21\frac{87}{93} = 21.935$.

(b) (i) $\frac{14}{21} + \frac{6}{21} = \frac{20}{21}$ (ii) $\frac{5}{6} - \frac{4}{6} = \frac{1}{6}$ (iii) $\frac{16}{60} + \frac{21}{60} = \frac{37}{60}$

(iv) $\frac{17}{5} - \frac{23}{8} = \frac{136}{40} - \frac{115}{40} = \frac{21}{40}$ (v) $\frac{3}{8} \times \frac{16}{11} = \frac{6}{11}$

(vi) $\frac{7}{4} \div \frac{14}{3} = \frac{7}{4} \times \frac{3}{14} = \frac{3}{8}$

Solution 7

Did you use the reciprocal button (marked $1/x$ or x^{-1}) on your calculator for parts (d) and (e)?

See Section 1.4.

(a) 0.1 (b) 0.2 (c) 10 (d) 0.0138 (to 3 s.f.)

(e) 281 (to 3 s.f.)

Solution 8

See Section 1.4.

(a) (i) 10% of 14 is 1.4, so a 10% increase gives $14 + 1.4 = 15.4$.

(ii) 15% of 142 is 21.3, so a 115% increase is $142 + 21.3 = 163.3$. The total is therefore $142 + 163.3 = 305.3$.

(b) (i) The actual increase is £17.50; the percentage increase is $17.5/300\% = 5.83\%$ (to 2 d.p.).

(ii) The actual decrease is £240; the percentage decrease is $240/300\% = 80\%$.

Solution 9

See Section 1.5.

(a) (i) $3 + (-4) - 6 = 3 - 4 - 6 = -7$

(ii) $3 - 4 + (-6) = 3 - 4 - 6 = -7$

(iii) $3 + (-4 + 6) = 3 + 2 = 5$

(iv) $3 \times (-4 + 6) = 3 \times 2 = 6$

(v) $(-6) \div (-5) = (-6)/(-5) = 6/5 = 1.2$

(vi) $6 \div (3 \div 5) = 6 \div \frac{3}{5} = 6 \times \frac{5}{3} = 10$

Your calculator probably gave different answers for some of the calculations.

See Section 1.7.

(b) (i) $\sqrt{14\,400} = \sqrt{12^2 \times 10^2} = 12 \times 10 = 120$

(ii) $\sqrt{0.81} = \sqrt{\frac{81}{100}} = \sqrt{\frac{9^2}{10^2}} = \frac{9}{10} = 0.9$

(iii) $12 = 2^2 \times 3$, so $\sqrt{12} = \sqrt{2^2 \times 3} = 2\sqrt{3}$.

(iv) $24 = 2^3 \times 3$, so $\sqrt[3]{24} = \sqrt[3]{2^3 \times 3} = 2\sqrt[3]{3}$.

(v) $10\,000 = 100^2$, so $\sqrt{\frac{1}{10\,000}} = \sqrt{\frac{1}{100^2}} = \frac{1}{100} = 0.01$.

(The answers to parts (iii) and (iv) have been left in 'surd' form, but could be expressed in decimal form with the aid of a calculator.)

Solution 10

(a) $\log_{10} 10^{-2} = -2$ (b) $\log_{10} \dfrac{1}{100} = -2$ (c) $\log_{10} 0.001 = -3$ See Section 1.6.

Solution 11

If you have difficulty in obtaining the answer shown, check your rounding. See Section 1.6.

(a) (i) 1.1449 (ii) 2.6504 (iii) 9.5927 (iv) -3.4296

(b) (i) 2.25 (ii) 5.07 (iii) 19.6 (iv) 1.04

Solution 12

The total of the ratios is $2 + 3 + 5 = 10$. One tenth of £70 is £7, so the first person pays $2 \times £7 = £14$, the second pays $3 \times £7 = £21$ and the third pays $5 \times £7 = £35$. See Section 1.8.

Solution 13

(a)

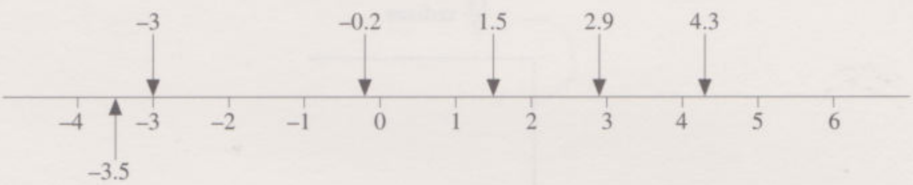


Figure 0.8

So, arranged in ascending order, the numbers are See Section 1.1.

$$-3.5,\quad -3,\quad -0.2,\quad 1.5,\quad 2.9,\quad 4.3.$$

(b) (i) $2.9 < 4.3$ (ii) $-0.2 < 1.5$ (iii) $2.9 > 1.5$ (iv) $-0.2 > -3$ See Section 2.2.

Solution 14

(a) $x = \dfrac{y-3}{2}$ or $x = \dfrac{1}{2}(y-3)$ See Section 2.3.

(b) $x^2 = p - 3$, so $x = \pm\sqrt{p-3}$ (provided $p \geq 3$)

(c) $x + 3 = \pm\sqrt{t}$, so $x = -3 \pm \sqrt{t}$ (provided $t \geq 0$)

(d) $2y = \sqrt{x}$, so $x = (2y)^2 = 4y^2$

(e) Multiply through by x^2 to give

$$mx^2 = 3p,$$

from which

$$x^2 = \frac{3p}{m}.$$

So $x = \pm\sqrt{\dfrac{3p}{m}}$ (provided $p/m \geq 0$).

Solution 15

See Section 2.4.

(a) One complete turn $= 2\pi$ radians $(= 360°)$.

(b)

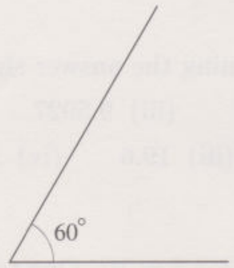


Figure 0.9

$$60° = \frac{60}{360} \times 2\pi \text{ radians} = \frac{\pi}{3} \text{ radians}$$

(c)

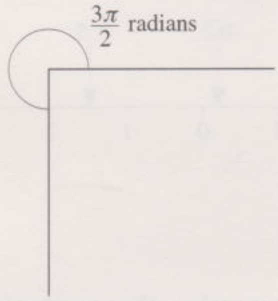


Figure 0.10

$$\frac{3\pi}{2} \text{ radians} = \frac{3\pi}{2} \times \frac{360°}{2\pi} = 270°$$

Solution 16

See Section 2.5.

(a) $(50 + 47 + 48 + 51 + 51) \div 5 = 49.4$, so the mean is 49.4 matches. The median is the middle value when the values have been arranged in order of magnitude, so is 50 matches. The mode is the value that occurs most often – that is, 51 matches.

(b) The mean is 49.2 matches. There are now ten values, so there is no single 'middle' value. The median is the mean of 48 and 50, the two 'middle' values – that is, 49 matches. The mode is 48 matches.

Solution 17

See Sections 3.1, 3.2, 3.4 and 6.2.

(a) $AC = \sqrt{(6^2 + 8^2)}\,\text{m} = \sqrt{36 + 64}\,\text{m} = \sqrt{100}\,\text{m} = 10\,\text{m}$.

(b) Perimeter of triangle $ABC = 6\,\text{m} + 8\,\text{m} + 10\,\text{m} = 24\,\text{m}$.

(c) Area of rectangle $= 6\,\text{m} \times 8\,\text{m} = 48\,\text{m}^2$, so area of triangle ABC is $24\,\text{m}^2$.

(d) Area of triangle $= \frac{1}{2} \times$ base $\times$ height, so

$$\text{area of triangle } ABC = \frac{1}{2} \times 8\,\text{m} \times 6\,\text{m} = 24\,\text{m}^2.$$

(e)

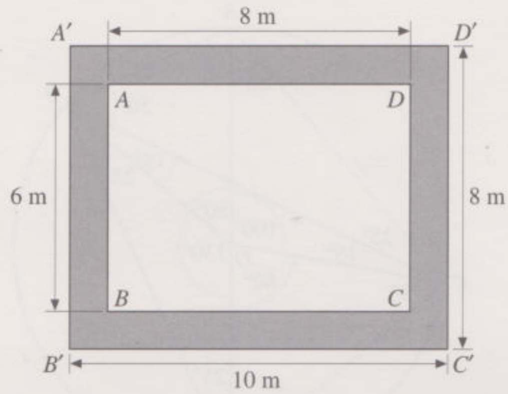


Figure 0.11

The diagram shows the measurements after the path, of width 1 m, has been added.

New rectangle $A'B'C'D'$ has area $10\,\text{m} \times 8\,\text{m} = 80\,\text{m}^2$.

Original rectangle $ABCD$ has area $48\,\text{m}^2$, from above.

So area of path $= 80\,\text{m}^2 - 48\,\text{m}^2 = 32\,\text{m}^2$.

Solution 18

See Sections 3.1, 3.3, 3.4 and 6.2.

(a) AD is a diameter and the angle in a semicircle subtended by a diameter is always 90°.

(b) (i) Isosceles triangles have two sides equal, so look for triangles formed by radii of the circle. Examples include triangles OBC, OAB, OBD, OCD.

(ii) Scalene triangles have no sides equal. Examples include triangles ABC, BCD, ACD, ABD.

(iii) Since the angle in a semicircle is 90°, ACD and ABD are right-angled triangles.

(c) Triangle AOC is isosceles ($AO = OC =$ radius), so

$$\text{angle } ACO = \text{angle } DAC = 40°.$$

Hence

$$\text{angle } AOC = 180° - (40° + 40°) = 100°.$$

AOD is a straight line, so

$$\text{angle } COD = 180° - 100° = 80°.$$

The other angles are shown in Figure 0.12, overleaf.

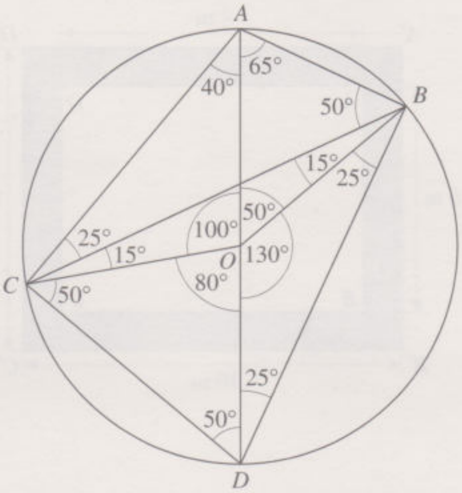


Figure 0.12

(d) (i) Area of triangle OCD is

$$\tfrac{1}{2} \times 10\,\text{cm} \times 10\,\text{cm} \times \sin 80^\circ = 49.24\,\text{cm}^2$$
$$= 49\,\text{cm}^2 \quad \text{(to nearest whole number).}$$

(ii) Length of arc CD is

$$\frac{80}{360} \times \text{circumference} = \frac{80}{360} \times 2\pi \times 10\,\text{cm}$$
$$= 14\,\text{cm} \quad \text{(to the nearest whole number).}$$

(e) Sector OCD has area $\frac{80}{360} \times \pi \times 10^2\,\text{cm}^2 = 70\,\text{cm}^2$ (to the nearest whole number).

Solution 19

See Sections 4.1 and 4.2.

(a) A is $(2, 3)$, B is $(3, -2)$ and C is $(-1, -2)$.

(b)

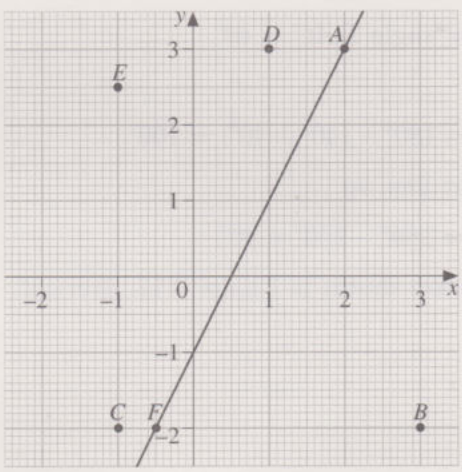


Figure 0.13

(c) See Figure 0.13.

(i) The gradient of the line through A and F is $\dfrac{3-(-2)}{2-(-0.5)} = \dfrac{5}{2.5} = 2$.

(ii) The line crosses the x-axis at $(\frac{1}{2}, 0)$.

(iii) The line crosses the y-axis at $(0, -1)$.

Solution 20

If your answers are wrong in this section, do not be disheartened; just try to identify your mistakes from the working shown here.

See Section 5.2.

(a) (i) $5 \times (-8) = -40$

(ii) $(-5) \times (-8) = 40$

(iii) $3 \times (-8) - 2 = -24 - 2 = -26$

(iv) $3(-8+2) = 3 \times (-6) = -18$

(b) (i) $(-8) \times (-8) + 4 = 64 + 4 = 68$

(ii) $(-8+4)^2 = (-4)^2 = 16$

(iii) $(-8+24)^{\frac{1}{2}} = 16^{\frac{1}{2}} = \sqrt{16} = +4$

(iv) $(-8)^2 - 3(-8) - 4 = 64 + 24 - 4 = 84$

Solution 21

(a) $2a - 4a = -2a$, so $2a + 3b - 4a = 3b - 2a$.

See Section 5.2.

(b) $2a + 3b - 4c - (a - 5b) = 2a + 3b - 4c - a + 5b = a + 8b - 4c$

(c) $2a + 3b - 4c + 2(a - 5b) = 2a + 3b - 4c + 2a - 10b = 4a - 7b - 4c$

(d) $2a + 3b - 4c - 2(a - 5b) = 2a + 3b - 4c - 2a + 10b = 13b - 4c$

(e) $$\begin{aligned} 2a + 3b - 4c + c(2 - 5b) &= 2a + 3b - 4c + 2c - 5bc \\ &= 2a + 3b - 2c - 5bc \end{aligned}$$

Solution 22

(a) $b(b+2) + 1(b+2) = b^2 + 2b + b + 2 = b^2 + 3b + 2$

See Section 5.2.

(b) $c(c+5) - 2(c+5) = c^2 + 5c - 2c - 10 = c^2 + 3c - 10$

(c) $d(d-f) - f(d-f) = d^2 - df - fd + f^2 = d^2 - 2df + f^2$

(d) $3x(2x-7) + 4(2x-7) = 6x^2 - 21x + 8x - 28 = 6x^2 - 13x - 28$

(e) $$\begin{aligned} 3x(5y+6x) - 4y(5y+6x) &= 15xy + 18x^2 - 20y^2 - 24xy \\ &= 18x^2 - 9xy - 20y^2 \end{aligned}$$

Solution 23

(a) $a(b+1)$

See Section 5.2.

(b) $a(b+c)$

(c) $a(b+2c+a)$

(d) $(a-b)(a+b)$

(e) $(a+b)^2$

(f) $(3a-1)(2a+1)$

Solution 24

See Section 5.3.

(a) (i) $2x = 15 + 5$, so $2x = 20$ and $x = 10$.

(ii) $2x - 10 = 15$, so $2x = 25$ and $x = 12.5$.

(iii) $2(x-5)^2 = 32$, so $(x-5)^2 = 16$. Taking square roots gives $x - 5 = \pm 4$, from which $x = 5 + 4 = 9$ or $x = 5 - 4 = 1$.

The solutions are $x = 1$ and $x = 9$.

Alternatively, it is possible to multiply out and factorise the result, as follows.

$$(x-5)^2 = 16$$

becomes

$$x^2 - 10x + 25 = 16$$

so that

$$x^2 - 10x + 9 = 0,$$

from which

$$(x-9)(x-1) = 0.$$

So $x = 9$ or $x = 1$. The first method is easier, however.

(iv) $8 = x + 7 + 2x$, so $8 = 3x + 7$ giving $8 - 7 = 3x$. Hence $3x = 1$ and so $x = \frac{1}{3}$.

(b) (i) Using the formula for the solution of a quadratic equation gives

$$\begin{aligned} p = \frac{-2 \pm \sqrt{2^2 - 4(-4)}}{2} &= \frac{-2 \pm \sqrt{4+16}}{2} \\ &= \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2}. \end{aligned}$$

So $p = -1 \pm \sqrt{5}$, that is, $p = 1.24$ or $p = -3.24$ (to 2 d.p.).

(ii) Multiplying both sides of $36 = \dfrac{25}{v^2}$ by v^2 gives $36v^2 = 25$, from which

$$v^2 = \frac{25}{36}.$$

Taking square roots gives $v = \pm\frac{5}{6}$.

(iii) Because the unknown t is in the index position, it is necessary to take logs of both sides, which gives $t \log 5 = \log 26$. So

$$t = \frac{\log 26}{\log 5} = 2.02 \quad \text{(to 2 d.p.)}.$$

(It doesn't matter whether you use logs to base 10 or ln, provided you are consistent: the same answer will be obtained.)

Solution 25

There is more than one way of solving a pair of simultaneous equations.

First, we label the equations:

$$x + 2y = 4, \tag{0.1}$$
$$2x - 3y = -6. \tag{0.2}$$

From Equation (0.1), $x = 4 - 2y$. Then substituting for x in Equation (0.2) gives

$$2(4 - 2y) - 3y = -6,$$

so that

$$8 - 4y - 3y = -6,$$

from which

$$7y = 14.$$

Hence $y = 2$. Substituting this value in Equation (0.1) gives

$$x + 4 = 4,$$

so that $x = 0$.

The solution is $x = 0$, $y = 2$.

Solution 26

(a) $2x \geq 5 + 3$, so $2x \geq 8$, giving $x \geq 4$.

See Section 5.4.

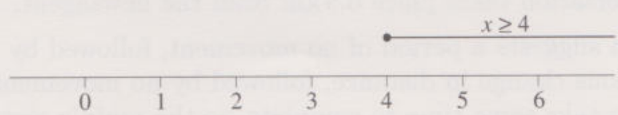


Figure 0.14

(b) $2x < 1 + 4$, so $2x < 5$, giving $x < 2.5$.

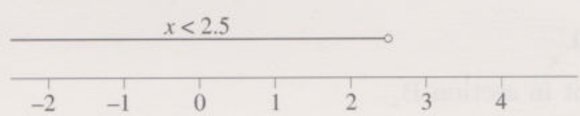


Figure 0.15

(c) Care is needed with this one!

$3 > 5 + x$, so $3 - 5 > x$, giving $-2 > x$, that is, $x < -2$.

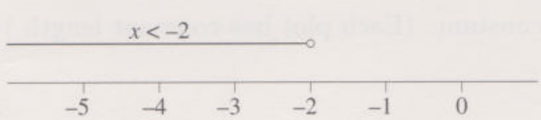


Figure 0.16

See Section 6.1.

Solution 27

The sum of the angles of the triangle is 180°, so the remaining angle is

$$180° - 90° - 70° = 20°.$$

In the triangle, BC is the hypotenuse, so

$$AB = BC \times \cos 70° = 15\,\text{cm} \times \cos 70° = 5.13\,\text{cm}$$

and

$$AC = BC \times \sin 70° = 15\,\text{cm} \times \sin 70° = 14.1\,\text{cm}.$$

(You can obtain the same answers by using different ratios of the angle 20°. Having obtained the length of one of the sides, you could have used Pythagoras' Theorem to calculate the length of the third side.)

Solution 28

See Section 7.1.

(a) (i) The distance from Chris's home to the newsagent is 0.9 km.

(ii) His speed in the first part of journey is 0.9 km in 10 minutes, which is 5.4 km/hour.

At the shop, the speed is 0 km/hour.

His speed in the first part of journey home is 0.5 km in 5 minutes, which is 6 km/hour.

During the conversation, the speed is 0 km/hour.

His speed in the final part of journey is 0.4 km in 5 minutes, which is 4.8 km/hour.

(iii) Conversation takes place 0.5 km from the newsagent.

(b) This graph suggests a period of no movement, followed by an instantaneous change in distance, followed by no movement. All movements take some time to complete, so the middle section is impossible.

Solution 29

See Section 7.1.

(a) For each plot in section A, the length times the breadth equals 500 m^2. In symbols,

$$lb = 500.$$

(b) For each plot in section B,

$$l = \tfrac{8}{5}b.$$

(c) (i) Graph I corresponds to $l = \frac{8}{5}b$. Graph III corresponds to $lb = 500$.

(ii) Graph II corresponds to a relationship in which the area increases in direct proportion to the breadth, that is,

$$\text{area} = kb,$$

where k is a constant. (Each plot has constant length.)

Solution 30

The appropriate graphs are given below.

See Section 7.2.

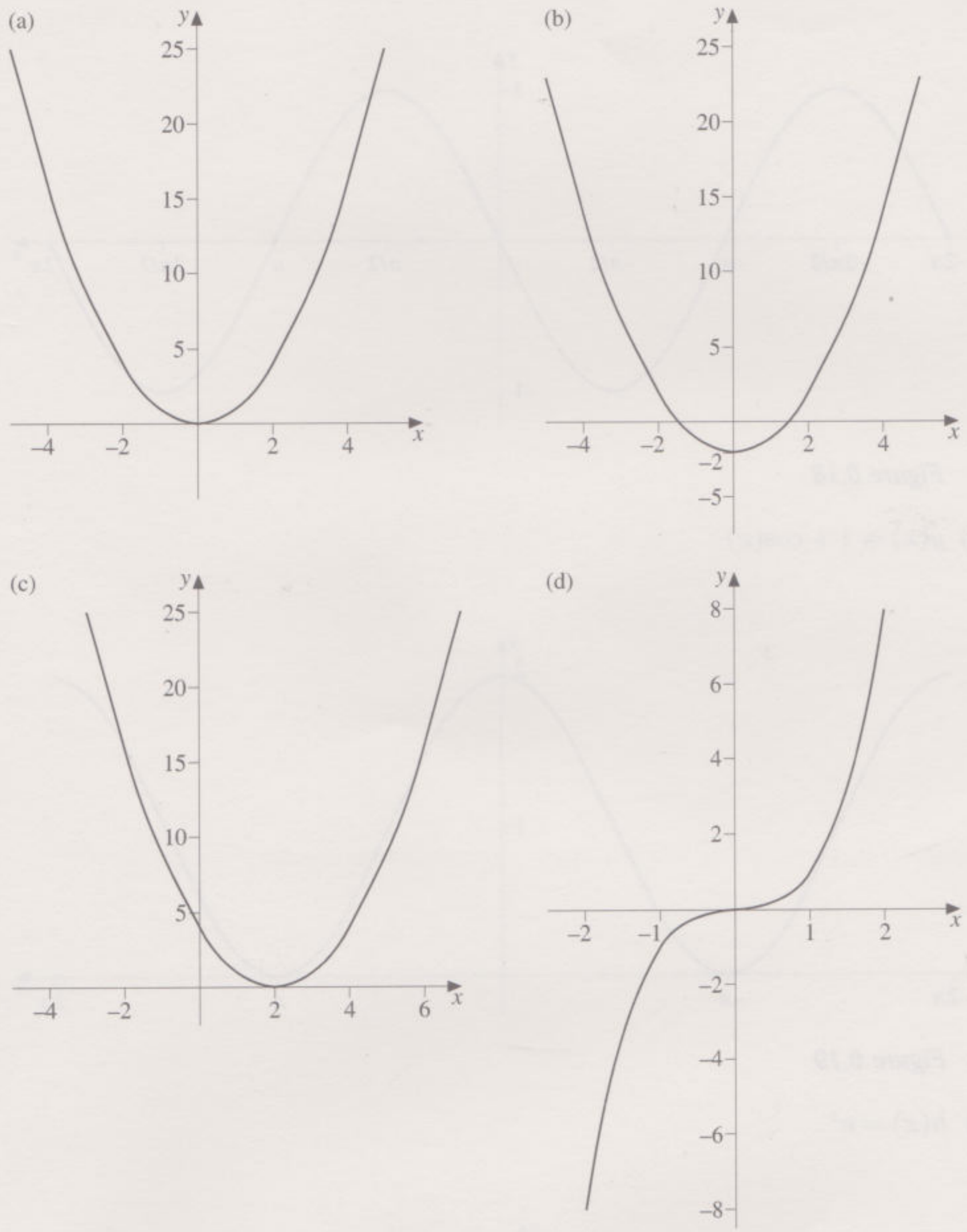


Figure 0.17

Solution 31

See Section 7.2.

The appropriate graphs are given below.

(a) $f(x) = \sin(x)$

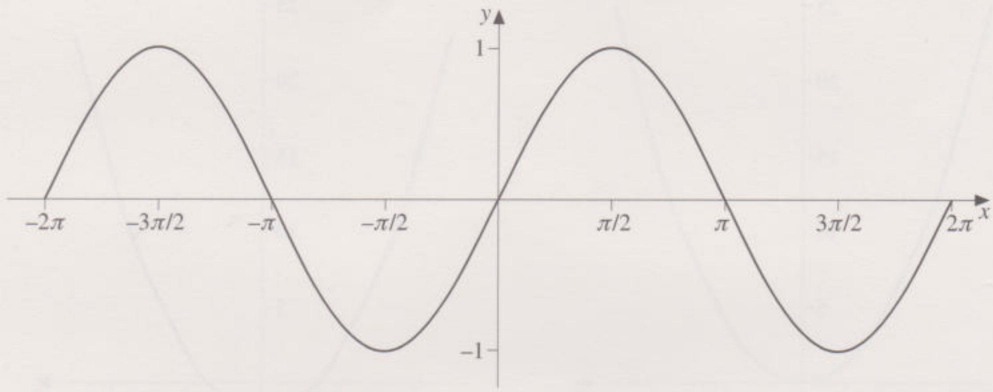


Figure 0.18

(b) $g(x) = 1 + \cos(x)$

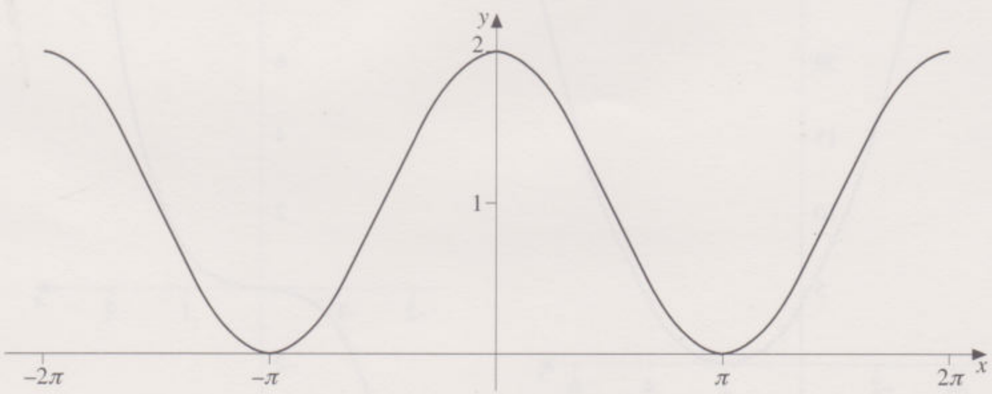


Figure 0.19

(c) $h(x) = e^x$

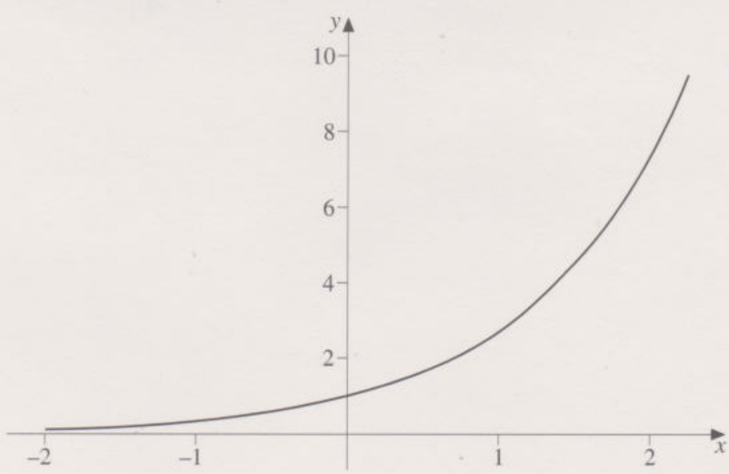


Figure 0.20

Solution 32

(a) (i) Let r be the radius of the circular hole. The circumference of the circle ($= 2\pi r$) is 30 cm, so

See Sections 8.1 and 8.2.

$$r = 30/2\pi = 4.77\,\text{cm}.$$

(This answer has been rounded to two decimal places for convenience, but the calculator gives more places. It is the unrounded version that should be used to find the area. Also make sure that you have used the π key on your calculator, and not an approximate value for π.)

Area of hole is $\pi r^2 = 71.62\,\text{cm}^2$ correct to two decimal places. (Using the rounded version for the area gives 71.48!)

(ii) Length of each side of the triangle is $30\,\text{cm}/3 = 10\,\text{cm}$.

$$\begin{aligned}\text{Area of triangle} &= \tfrac{1}{2} \times 10\,\text{cm} \times 10\,\text{cm} \times \sin 60^\circ \\ &= 50 \times \frac{\sqrt{3}}{2}\text{cm}^2 = 25\sqrt{3}\,\text{cm}^2.\end{aligned}$$

So area is $43.30\,\text{cm}^2$ (to 2 d.p.).

(iii) Since there is no top or bottom to either shape, the surface area of each is just the area of the paper used, minus the area of the one-centimetre seam. So surface area of each is

$$20\,\text{cm} \times 30\,\text{cm} = 600\,\text{cm}^2.$$

(iv) Volume of cylinder is

$$\text{area of circle} \times \text{length} = \pi r^2 \times \text{length}.$$

The length is 20 cm and $r = 30/2\pi$ from part (i). So volume of cylinder is

$$\pi \times \left(\frac{30}{2\pi}\right)^2 \text{cm}^2 \times 20\,\text{cm} = 1432.39\,\text{cm}^3 \quad \text{(to 2 d.p.)}.$$

Volume of prism is

$$\begin{aligned}\text{area of triangular base} \times \text{length} &= 25\sqrt{3}\,\text{cm}^2 \times 20\,\text{cm} \\ &= 866.03\,\text{cm}^3 \quad \text{(to 2 d.p.)}.\end{aligned}$$

So ratio of the volume of the cylinder to that of the prism is approximately

$$\frac{1432.39}{866.03} = 1.65.$$

(The ratio is an approximate one since the numbers quoted are rounded figures, not the full calculator accuracy.)

(b) There are many tubes in common usage; for example, the inside of toilet and kitchen rolls, the tubular part of a Smarties tube.
The prism has more limited use, although Toblerone is sold in prism-shaped boxes.

Module 1 Numbers

1.1 Number system and notation

Naming numbers

Over the centuries, as human activity became more complex, there was a need to develop the concept of number for different purposes. Initially the only need was for the counting of numbers:

one, two, three, ...

These positive whole numbers are now commonly referred to as the **natural numbers**. As commercial, scientific and mathematical activity developed, so did fractions, negative numbers, the concept of zero, and the various symbols, systems and notations for expressing them.

The negative and positive whole numbers and 0 (zero) are referred to as the **integers**, and can be visualised as lying on a **number line**.

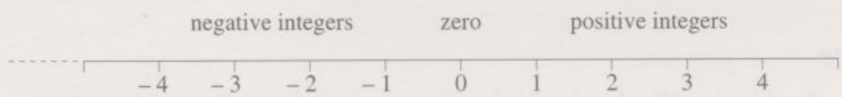


Figure 1.1

Fractions, also referred to as **rational numbers**, are obtained by dividing two integers and can be written as 'bar' fractions (often called common or vulgar fractions), such as $\frac{1}{2}$. Rational numbers include the integers because each integer can be rewritten as a bar fraction; for example, $2 = \frac{2}{1}$. All rational numbers can be written as **decimals** which either **terminate** – for example, $\frac{5}{4} = 1.25$ and $-\frac{4}{10} = -0.4$ – or **recur** – for example, $\frac{1}{3} = 0.333\,333\,333\ldots$ and $-\frac{8}{11} = -0.727\,272\,727\ldots$. (The shorthand for a recurring decimal is a dot above the recurring digit or at each end of a set of recurring digits; for example, $\frac{1}{3} = 0.\dot{3}$, $\frac{8}{11} = 0.\dot{7}\dot{2}$, and $\frac{41}{333} = 0.\dot{1}2\dot{3}$.)

Also $0 = \frac{0}{1}$, for example.

0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are the **digits**.

Figure 1.2 illustrates the composition of the set of rational numbers.

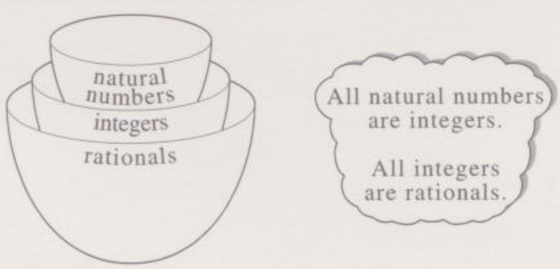


Figure 1.2

There are also infinite decimals that neither terminate nor recur, and these are called **irrational numbers**. They include numbers like $\sqrt{2} = 1.4142\ldots$, $-\sqrt{5} = -2.2360\ldots$, $\pi = 3.1415\ldots$ and $e = 2.7182\ldots$.

π is the Greek letter pi. e is the base for natural logarithms.

The set of numbers that includes the rationals and the irrationals is called the set of **real numbers**. (Square roots can be written as either $\surd 3$ or $\sqrt{3}$. In this text we use only the latter.)

Each real – rational or irrational – has its place on the number line. Some examples are shown in Figure 1.3. The gaps between the integers on the number line are completely filled by rational and irrational numbers.

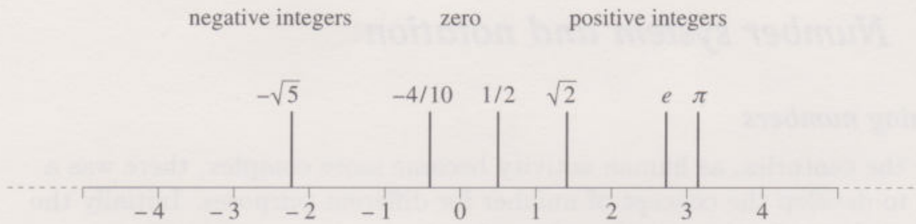


Figure 1.3

Powers and roots are explained in more detail later in this section.

A particular real number may be written in a variety of forms; for example, two (word), 2 (rational number), +2 (positive integer), 2.0 (decimal), $\frac{2}{1}$ or $\frac{4}{2}$ (fractions), 2^1 (power) and $\sqrt{4}$ (positive square root) represent the *same number*. Different forms have been developed to suit particular purposes.

Experiment to ensure that you can input the different forms of numbers on your particular calculator. For example, do you need to press function keys *before* or *after* you enter digits?

Place value

The Indian–Arabic (or Hindu–Arabic) system most generally used in the western world has ten numeral symbols (the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9) which are combined in a place-value system structure. The **place-value** system is based on powers of ten and extends to decimals. For example, the numeral 23 789.56 (twenty-three thousand seven hundred and eighty-nine point five six) has seven digits, the value of each of which is indicated in the following table.

	Ten thousands	Thousands	Hundreds	Tens	Units	tenths	hundredths
integer/ decimal	10 000	1000	100	10	1	0.1	0.01
power form	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}
example	2	3	7	8	9	5	6

Indices

Index notation is a concise way of symbolising the repeated multiplication of a number by itself. For example, multiplying 10s together successively gives:

$$10 \times 10 = 100, \quad 10 \times 10 \times 10 = 1000, \quad 10 \times 10 \times 10 \times 10 = 10\,000.$$

Using the shorthand notation, we write:

$$10^2 = 100, \quad 10^3 = 1000, \quad 10^4 = 10\,000, \quad \ldots .$$

In this example, 10 is the base number and the superscript number at the top right indicates how many of these base numbers have been multiplied together. This superscript number is variously called the **power**, **index** or **exponent**. This shorthand is extended as follows:

$$10^1 = 10, \quad 10^0 = 1, \quad 10^{-1} = \frac{1}{10} \text{ (or 0.1)},$$

$$10^{-2} = \frac{1}{10 \times 10} = \frac{1}{100} \text{ (or 0.01)}, \quad \ldots .$$

The notation also extends to fractional powers. For example:

$$10^{1/2} = \sqrt{10}, \quad 10^{-1/2} = \frac{1}{\sqrt{10}}.$$

A justification for assigning the value $\sqrt{10}$ to the symbol $10^{1/2}$ (or $10^{0.5}$) is given in Section 1.6.

Example 1.1

Evaluate each of the following powers, first by hand and then using your calculator.

(a) 5^3 (b) 10^6 (c) 2^{-3}

Solution

(a) $5^3 = 5 \times 5 \times 5 = 125$
5 [y^x] 3 [=] 125 (scientific calculator)
or 5 [$\wedge$] 3 [ENTER] 125 (graphics calculator)

Calculator keys are indicated by square brackets [ ]. The actual keys used vary from one make of calculator to another.

(b) $10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 1\,000\,000$
10 [y^x] 6 [=] 1000000
or 10 [$\wedge$] 6 [ENTER] 1000000

(c) $2^{-3} = \frac{1}{2\times2\times2} = \frac{1}{8}$ (or 0.125)
2 [y^x] 3 [+/−] [=] 0.125
or 2 [$\wedge$] [(−)] 3 [ENTER] .125

Exercise 1.1

(a) Evaluate each of the following 'by hand'.

(i) 10^4 (ii) 10^5 (iii) 3^4

(b) Use your calculator to find each of the following.

(i) 6^5 (ii) 4.3^4 (iii) $(-7.1)^3$ (iv) $(-8)^7$ (v) 12^4

Scientific notation

Any real number can be expressed in the form

(number between 1 and 10, but not including 10) × (power of ten).

For example:

$$\begin{aligned}253 &= 2.53 \times 100 &&= 2.53 \times 10^{2},\\ 25.3 &= 2.53 \times 10 &&= 2.53 \times 10^{1},\\ 2.53 &= 2.53 \times 1 &&= 2.53 \times 10^{0},\\ 0.253 &= 2.53 \times 0.1 &&= 2.53 \times 10^{-1},\\ 0.0253 &= 2.53 \times 0.01 &&= 2.53 \times 10^{-2}.\end{aligned}$$

This way of writing a number indicates its order of magnitude as the exponent, and is useful when calculating using very large or very small numbers, and particularly so when dealing with a mixture of large and small numbers. It is commonly referred to as **scientific notation** but is also known as **standard form**. It is this notation which scientific calculators use to show very large and very small numbers – sometimes the power is indicated by the letter E (for exponent). For example, 2.53E–1 means 0.253.

Exercise 1.2

(a) Express each of the following numbers in scientific notation.

(i) 1 427 000 000 (ii) 8075 (iii) 0.003 27 (iv) 0.5672
(v) 0.000 000 400 7

(b) Express each of the following numbers in full.

(i) 3.298×10^{5} (ii) 7.654×10^{1} (iii) 1.098×10^{-3}
(iv) 3.4×10^{-10}

1.2 Calculating

Order of operations

Imagine that someone asks you to do the following calculations (read them out loud with ... ... being a dramatic pause).

'What is 2 add 3 ... ... times 4?'
'What is 2 ... ... add 3 times 4?'

The first is likely to have produced the answer 20, the second 14.

Since there are no dramatic pauses in written calculations (or on calculators or computers), there is a universally accepted convention to overcome this and similar possible misunderstandings.

Scientific calculators follow this convention but most non-scientific ones do not.

In calculations, operations are performed in the following order.

- ◇ **B**rackets
- ◇ **I**ndices
- ◇ **D**ivision and **M**ultiplication (the order does not matter)
- ◇ **A**ddition and **S**ubtraction (the order does not matter)

One way of remembering this is the mnemonic **BIDMAS**.

Exercise 1.3

(a) Insert brackets in each of the following calculations where necessary to emphasise the order in which it must be performed according to the above convention. Then do the calculations without using a calculator.

(i) $3 + 5 \times 2$ (ii) $10^3 \times 3$ (iii) $\dfrac{15+5}{3+7}$ (iv) $6 - 4 + 2$

(v) $2^2 + 3 \times 10^2$

(b) What happens when you use your calculator without the brackets?

Process

For any mathematical problem requiring an extensive calculation there are five stages.

1 Establish the calculation to be done.

2 Make an **estimate** (particularly advisable when using a calculator).

3 Do the full calculation, using a calculator or computer if appropriate.

4 Verify the solution; first compare your exact answer against the estimate, to ascertain whether it is about the right size; then do some kind of check (for example, putting the solution back into the original problem, or redoing the calculation a different way).

5 Check that the answer makes sense in the context of the original problem. Round as necessary.

Example 1.2

A bus company owns a number of 42-seater coaches and has 185 passengers wishing to go to London. How many coaches are needed?

Solution

Stage 1 $185 \div 42$

Stage 2 Approximate to 'easy' numbers to obtain an estimate:

$200 \div 40 = 5$

Stage 3 $185 \div 42 = 4.4047\ldots$ (using a calculator)

Stage 4 Calculator answer is not too far from estimate, and

calculator answer $\times\, 42 = 185$

(so there were no errors keying in)

Stage 5 Fractions of a coach are not 'sensible', so round up to nearest whole number. Answer: 5 coaches.

Rounding to the nearest integer is not appropriate in this case, since the result would leave some passengers stranded.

Estimating

An estimate is a rough answer produced by using **approximate** numbers. Estimating answers to numerical calculations is important as it provides a way of checking that the final answer is of the correct order of magnitude, whether it is produced by 'hand' or using a calculator. The process involves substituting 'easy' numbers that can be worked on mentally or quickly using paper and pencil.

Example 1.3

Estimate the answers to the following calculations; then find the exact answers using a calculator.

(a) $44\,281 + 3729$ (b) $7002 - 633$ (c) 379.42×0.23

(d) $0.046 \div 1.5$

Solution

The symbol $\simeq$ is read as 'is approximately equal to'.

(a) $44\,281 \simeq 44\,000$ and $3729 \simeq 4000$; so an approximate answer is 48 000. Using a calculator gives the exact answer: $44\,281 + 3729 = 48\,010$.

(b) $7002 \simeq 7000$ and $633 \simeq 600$; so an approximate answer is 6400. Using a calculator gives $7002 - 633 = 6369$.

(c) $379.42 \simeq 400$ and $0.23 \simeq 0.2$; so an approximate answer is 80. (You might have said that $0.23 \simeq 0.25$, giving an approximate answer of 100.) Using a calculator gives $379.42 \times 0.23 = 87.2666$.

(d) $0.046 \div 1.5 = 0.46 \div 15 \simeq 0.45 \div 15 = 0.03$. Using a calculator which displays 10 digits gives $0.046 \div 1.5 = 0.030\,666\,666\,7$.

Exercise 1.4

Carry out each of the following calculations using your calculator, having first made an estimate of the answer you expect.

(a) 441.7×5.2 (b) $53.4 \times 70.9 \div 22.2$ (c) $217.5 + 60.3 \times 17.7$

(d) $(1285 - 329) \times 0.023$

Note the use of brackets in part (d) to indicate that the subtraction is to be performed before the multiplication.

Rounding

If the final answer to a numerical calculation involves a number with a large number of digits, it is often appropriate to give a rounded answer. Unless the nature of a particular problem determines whether a sensible answer is obtained by rounding down or up, a convention on how to round numbers is used.

In Example 1.2, 4.4047... was rounded up to 5, not down to 4. Use of the convention is illustrated in Example 1.4.

To round to a **given number of decimal places**:

◇ look at the digit which is one more place to the right of the number of places;

◇ round up if this digit is 5 or more, and down otherwise.

To round to a **given number of significant figures**:

◇ look at the digit which is that number of places to the right of the first non-zero digit;

◇ round up if this digit is 5 or more, and down otherwise.

Example 1.4

(a) Round $0.030\,666\,666\,7$ to four decimal places.

(b) Round $47.137\,253\,6$ to the following numbers of decimal places.

(i) two

(ii) three

(iii) four

(c) Round each of the following to two significant figures.

(i) $2\,295\,673$ (ii) $0.027\,294$ (iii) $0.000\,040\,510\,8$

Solution

(a) The digit in the fifth decimal place in $0.030\,6\underline{6}6\,666\,7$ is 6, which is more than 5, so the number is rounded up, giving
$0.030\,666\,666\,7 = 0.0307$ (to 4 d.p.).

For convenience the words 'decimal places' and 'significant figures' are often abbreviated to d.p. and s.f. respectively.

(b) (i) The digit in the third decimal place in $47.13\underline{7}\,253\,6$ is 7, so
$47.137\,253\,6 = 47.14$ (to 2 d.p.).

(ii) The digit in the fourth decimal place in $47.137\,\underline{2}53\,6$ is 2, so
$47.137\,253\,6 = 47.137$ (to 3 d.p.).

(iii) The digit in the fifth decimal place in $47.137\,2\underline{5}3\,6$ is 5, so
$47.137\,253\,6 = 47.137\,3$ (to 4 d.p.).

(c) (i) The digit two places to the right of the first non-zero digit, from the left, in $2\,2\underline{9}5\,673$ is 9, so $2\,295\,673 = 2\,300\,000$ (to 2 s.f.).

(ii) The digit two places to the right of the first non-zero digit, from the left, in $0.027\,\underline{2}94$ is 2, so $0.027\,294 = 0.027$ (to 2 s.f.).

(iii) The digit two places to the right of the first non-zero digit, from the left, in $0.000\,040\,\underline{5}10\,8$ is 5, so $0.000\,040\,510\,8 = 0.000\,041$ (to 2 s.f.).

Exercise 1.5

(a) Round $2.141\,599\,6$ to the following numbers of decimal places.

(i) one (ii) three (iii) four (iv) five

(b) (i) Round $63\,056$ to two significant figures.

(ii) Round 0.038 to one significant figure.

(iii) Round $0.040\,06$ to three significant figures.

(c) (i) Round 23.009 to one decimal place.

(ii) Round 9999 to three significant figures.

(iii) Round 6080 to two significant figures.

(iv) Round 16.99 to one decimal place.

1.3 Multiples, factors and prime numbers

With the advent of calculators, the use of **multiples** and **factors** in dealing with whole numbers is declining, but an understanding and facility with multiples and factors remains very important for algebra.

Multiples

The multiples of a positive whole number are those positive whole numbers into which it divides exactly: for example, the multiples of 6 are 6, 12, 18, 24, ...; that is, the multiples of a number are that number multiplied by 1, 2, 3, 4, ... However, finding the common multiples of two or more numbers is more difficult, and to find the **lowest common multiple (LCM)** needs a knowledge of factors.

Factors

The process of finding all the whole numbers that divide exactly into a given whole number is called 'finding the **factors**'. 'Factorising' means writing a number as the product of two or more whole numbers. For example, $48 = 12 \times 4$ and $48 = 8 \times 3 \times 2$.

The first six prime numbers are 2, 3, 5, 7, 11 and 13.

A **prime number** is a positive whole number, other than 1, that is divisible only by itself and 1. Factorising a number so that each factor is a prime number is often very useful – this process is called 'finding the prime factors'. For example, the factors of 48 which are prime are 2 and 3; the prime factors of 48 are

$$48 = 2 \times 2 \times 2 \times 2 \times 3 \; (= 2^4 \times 3).$$

The procedure for finding the prime factors of a number a are as follows. It is illustrated in Example 1.5(b) for $a = 60$.

To find the prime factors of a

Divide a by 2 (the first prime number) repeatedly until the result is not divisible by 2. If the number of such divisions is N, then 2^N is a factor of a.

If the result is a prime number, the process is complete. If not, repeat the process for each successive prime number, replacing 2 by 3, 5, 7, ..., as necessary.

Example 1.5

(a) Find the factors of 60.

(b) Find the prime factors of 60.

Solution

(a) The question is asking what pairs of whole numbers when multiplied together make 60. They are

$$1 \times 60, \quad 2 \times 30, \quad 3 \times 20, \quad 4 \times 15, \quad 5 \times 12, \quad 6 \times 10.$$

Thus the factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60.

(b) Divide 60 by 2: $60 = 2 \times 30$. 30 is divisible by 2.
Divide 30 by 2: $30 = 2 \times 15$. 15 is not divisible by 2.
Divide 15 by 3: $15 = 3 \times 5$. 5 is prime.

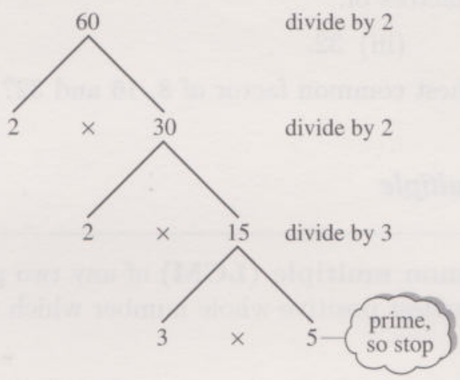


Figure 1.4

So the prime factors of 60 are 2, 2, 3, 5. Thus

$$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5.$$

With small numbers, you may be able just to write down the prime factors, without using the procedure.

Exercise 1.6

(a) Find the factors of 36.

(b) Find the prime factors of 36.

Highest common factor

The **highest common factor (HCF)** of any two positive whole numbers is the largest number which is a factor of both.

Or put another way, the HCF is the product of the lowest power of each prime factor common to both.

This definition extends easily to three or more numbers.

The following example illustrates how to find the HCF.

Example 1.6

What is the highest common factor of the numbers 18 and 30?

Solution

First, write each number in terms of its prime factors. We have

$$18 = 2 \times 3^2 \qquad \text{and} \qquad 30 = 2 \times 3 \times 5.$$

Taking the lowest power of the common prime factors gives

$$\text{HCF of 18 and } 30 = 2 \times 3 = 6.$$

Exercise 1.7

(a) Find the prime factors of:

(i) 8 (ii) 16 (iii) 32.

(b) What is the highest common factor of 8, 16 and 32?

Lowest common multiple

Again, this definition extends to three or more numbers.

> The **lowest common multiple (LCM)** of any two positive whole numbers is the smallest positive whole number which is a multiple of both.
>
> Or put another way, the LCM is the product of the highest power of each prime factor occurring.

Example 1.7

What is the lowest common multiple of each of the following sets of numbers?

(a) 10, 25. (b) 8, 24, 60.

Solution

(a) In terms of prime factors 10 can be written as 2×5 and 25 as $5 \times 5 = 5^2$. The LCM of 10 and 25 is the product of the highest power of each prime factor, that is

$$2 \times 5^2 = 2 \times 25 = 50.$$

So the LCM of 10 and 25 is 50.

(b) $8 = 2^3$.

$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3.$

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3 \times 5.$

So the LCM of 8, 24, and 60 is $2^3 \times 3 \times 5 = 120$.

Exercise 1.8

(a) What is the LCM of 28 and 36?

(b) What is the LCM of 7, 10 and 14?

1.4 Fractions, decimals and percentages

Some real numbers can be expressed as exact **fractions**, that is, as one whole number divided by another whole number, such as:

$$\tfrac{3}{5}\ (= 3 \div 5); \quad 2\tfrac{1}{4} = \tfrac{9}{4}\ (= 9 \div 4); \quad -\tfrac{2}{7}\ (= -2 \div 7, \text{ or equally } 2 \div -7.)$$

The number written above the line or bar in a fraction is called the **numerator**, and the number written below the line is called the **denominator**.

Converting fractions into decimals

Fractions are converted into decimals by dividing the numerator by the denominator.

Example 1.8

Convert $\frac{3}{5}, 2\frac{1}{4}$ and $-\frac{2}{7}$ into decimals.

Solution

(a) $\frac{3}{5} = 3 \div 5 = 0.6$

(b) $2\frac{1}{4} = \frac{9}{4} = 9 \div 4 = 2.25$

(c) $-\frac{2}{7} = -(2 \div 7) = -0.285\,714\,285\,714\ldots$

In (a) and (b) above, the division process **terminates**.

In (c) above, the division process does not terminate but goes into a repeating loop that generates the digits 285714 over and over again. The decimal for $-\frac{2}{7}$ is said to be **recurring**, and can be written as $-0.\dot{2}85\,71\dot{4}$, where the two dots above the digits 2 and 4 indicate the block of digits that is repeated.

If the result of a calculation is a final answer to a question, it can be rounded by making a statement like: $-\frac{2}{7} = -0.2857$ to four decimal places. Numbers which are going to be used in subsequent calculations should not be rounded.

Exercise 1.9

Convert each of the following fractions into decimals.

(a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $3\frac{3}{4}$ (d) $1\frac{1}{8}$ (e) $3\frac{3}{7}$

Equivalent fractions

When the numerator and denominator of a fraction are both multiplied or divided by the same whole number (other than 0), the new fraction obtained is equal to the original one, and the two fractions are called **equivalent fractions**. They are represented by the same point on the number line.

Example 1.9

Write down the fractions equivalent to $\frac{3}{5}$ which are obtained by multiplying its numerator and denominator by each of the following numbers.

(a) 2 (b) 3 (c) 10 (d) 20 (e) -1

Solution

(a) $\dfrac{3 \times 2}{5 \times 2} = \dfrac{6}{10}$ (b) $\dfrac{3 \times 3}{5 \times 3} = \dfrac{9}{15}$ (c) $\dfrac{3 \times 10}{5 \times 10} = \dfrac{30}{50}$

(d) $\dfrac{3 \times 20}{5 \times 20} = \dfrac{60}{100}$ (e) $\dfrac{3 \times -1}{5 \times -1} = \dfrac{-3}{-5}$

Any whole number that divides into each of two numbers is called a **common factor** of those numbers.

In a complete set of equivalent fractions, the fraction with the smallest possible positive denominator is said to be the **fraction in its lowest terms**. This is found from any one of the equivalent fractions by **cancelling**, that is, by successive division of the numerator and denominator by each of their common prime factors. (This is equivalent to division by the HCF of the numerator and denominator.)

Example 1.10

Express (a) $\frac{28}{42}$ and (b) $\frac{360}{240}$ as fractions in their lowest terms.

Solution

(a) We have $28 = 2 \times 2 \times 7$ and $42 = 2 \times 3 \times 7$, so the common prime factors are 2 and 7. Hence

$\frac{28}{42} = \frac{4}{6}$ (dividing by 7)

$= \frac{2}{3}$ (dividing by 2).

(b) $\frac{360}{240} = \frac{36}{24}$ (dividing by $10 = 2 \times 5$)

$= \frac{3}{2}$ (dividing by $12 = 2 \times 2 \times 3$)

In practice, spotting 'large' common factors, as here, is an efficient way to proceed.

Calculations involving fractions

Adding and subtracting fractions

Two fractions are added (or subtracted) by converting each of them into equivalent fractions with the same denominator, adding (or subtracting) these equivalent fractions and, where appropriate, converting the result to a fraction in its lowest terms.

Example 1.11

Evaluate each of the following and express the result in its lowest terms.

(a) $\frac{1}{2}+\frac{1}{3}$ (b) $\frac{2}{3}-\frac{1}{4}$ (c) $\frac{1}{5}+\frac{3}{10}$

Solution

(a) $\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$

Notice that in this case the denominator of the equivalent fractions is the product of the denominators of the original fractions.

(b) $\frac{2}{3}-\frac{1}{4}=\frac{8}{12}-\frac{3}{12}=\frac{5}{12}$

(c) $\frac{1}{5}+\frac{3}{10}=\frac{2}{10}+\frac{3}{10}=\frac{5}{10}=\frac{1}{2}$

In this solution the equivalent fraction denominator used is the lowest common multiple of the denominators of the original fractions. However, the same final result would be obtained by using the product of the denominators:

$$\frac{1}{5}+\frac{3}{10}=\frac{10}{50}+\frac{15}{50}=\frac{25}{50}=\frac{1}{2}.$$

The reason for choosing to use lowest common denominators is to keep the numbers involved in calculating the equivalent fractions small enough to be done mentally quickly and easily.

Multiplying fractions

> The product of two fractions is obtained by multiplying their numerators and multiplying their denominators, to give a new numerator and denominator respectively, then, where appropriate, converting the result to a fraction in its lowest terms.

Example 1.12

Evaluate each of the following and express the result in its lowest terms.

(a) $\frac{1}{2}\times\frac{2}{3}$ (b) $\frac{3}{4}\times\frac{5}{8}$

Solution

(a) $\frac{1}{2}\times\frac{2}{3}=\frac{2}{6}=\frac{1}{3}$

(b) $\frac{3}{4}\times\frac{5}{8}=\frac{15}{32}$

The next example shows how to deal with division involving fractions.

Example 1.13

Evaluate each of the following and express the result in its lowest terms.

(a) $4\div\frac{1}{2}$ (b) $\frac{1}{2}\div\frac{1}{4}$ (c) $\frac{2}{3}\div\frac{1}{3}$ (d) $\frac{4}{5}\div\frac{2}{3}$

Solution

(a) $4 \div \frac{1}{2}$

This question is the same as 'how many halves are there in 4?'. The answer is 8 as there are 2 halves in 1, so there will be 4 times 2 halves in 4.

That is, $4 \div \frac{1}{2}$ is the same as $4 \times \frac{2}{1} = 8$.

(b) $\frac{1}{2} \div \frac{1}{4}$

In words, this question is asking 'how many quarters in one half?'. The answer is 2.

$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2.$$

(c) $\frac{2}{3} \div \frac{1}{3} = \frac{2}{3} \times \frac{3}{1} = \frac{6}{3} = 2.$

(d) $\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2} = \frac{12}{10} = \frac{6}{5}.$

An alternative way of simplifying a product of fractions is to employ 'cross cancellation'. For example,

$$\begin{aligned}\frac{4}{5} \div \frac{2}{3} &= \frac{4}{5} \times \frac{3}{2} \\ &= \frac{2}{5} \times \frac{3}{1} \quad \text{(here 2 has been divided into 4 and 2)} \\ &= \frac{6}{5}.\end{aligned}$$

Dividing fractions

> To divide a number by a fraction, multiply the number by the fraction turned 'upside down'. Another name for this 'upside down number' is 'reciprocal', a term which is explained more fully in the next subsection.

Mixed numbers

A **mixed number** is made up of a whole number and a fraction. For example, the number $2\frac{1}{2}$ is a mixed number. It can be converted into the equivalent improper or top-heavy fraction (where the numerator is larger than the denominator) as follows:

$$2\tfrac{1}{2} = 2 + \frac{1}{2} = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}.$$

The answer in Example 1.13(d) was in the form of an improper fraction: $\frac{6}{5}$. An improper fraction can be converted into a mixed number by dividing the top by the bottom to find the number of whole ones; the remainder gives the number of fraction parts left over.

For example, $\frac{7}{5} = 1\frac{2}{5}$ $(7 \div 5 = 1$, remainder $2)$.

The methods for calculating with mixed numbers are exactly the same as those for the methods given in the previous examples for proper fractions (in which the numerator is smaller than the denominator) provided that you change the mixed fraction to an improper fraction first.

Exercise 1.10

(a) Evaluate each of the following and express the result in its lowest terms.

(i) $\frac{2}{3}+\frac{1}{6}$ (ii) $1\frac{3}{4}+\frac{3}{8}$ (iii) $\frac{18}{25}-\frac{2}{5}$ (iv) $4\frac{2}{7}-1\frac{3}{5}$

(b) Evaluate each of the following and express the result in its lowest terms.

(i) $\frac{4}{5}\times\frac{2}{7}$ (ii) $\frac{4}{5}\div\frac{2}{7}$ (iii) $1\frac{2}{25}\times 3\frac{8}{9}$ (iv) $4\frac{2}{7}\times 1\frac{2}{3}$
(v) $1\frac{2}{3}\div 1\frac{1}{14}$

Reciprocals

The word **reciprocal** is the mathematical term for how many of a particular number are in 1. The reciprocal of $\frac{1}{2}$ is therefore 2 as there are 2 halves in 1.

The term applies to real numbers, but initially it helps to think in terms of positive whole numbers.

The reciprocal of a number is 1 divided by that number. For example, the reciprocal of 10 is $\frac{1}{10}$. In effect the number is inverted, because 10 when written as a rational number in lowest terms is $\frac{10}{1}$.

So the reciprocal of 10 can be written in fraction form as $\frac{1}{10}$ or in power form as 10^{-1}. As a decimal, its value is 0.1.

There is a key for finding reciprocals on your calculator though the answer is given as a decimal rather than a fraction. Try this on your calculator now for the number 10 by putting in 10 and then pressing the reciprocal key, which is marked $[1/x]$ or $[x^{-1}]$. With some calculators the answer 0.1 will be displayed immediately, with others you may need to press [ENTER]. Leave the 0.1 in the display on your calculator and then press the reciprocal key again. You will see that the reciprocal of 0.1 ($\frac{1}{10}$) is 10. The inverse operation of 'finding a reciprocal' is the same operation, i.e. 'finding a reciprocal'; this is an unusual result. It means that a number multiplied by its reciprocal always equals 1.

In the case of 10: $10\times\frac{1}{10}=1$.
In the case of $\frac{1}{10}$: $\frac{1}{10}\times 10=1$.
In the case of 2: $2\times\frac{1}{2}=1$.
In general: $n\times\dfrac{1}{n}=1$.

Example 1.14

Write down the reciprocal of each of the following numbers, giving the answer in both fraction and decimal form.

(a) 5 (b) $\frac{1}{4}$ (c) -100 (d) 3

Solution

(a) $\frac{1}{5}=0.2$ (b) $\frac{4}{1}=4$ (c) $\frac{1}{-100}=-\frac{1}{100}=-0.01$ (d) $\frac{1}{3}=0.\dot{3}$

Exercise 1.11

Write down the reciprocal of each of the following numbers, giving the answer in both fraction and decimal form.

(a) 8 (b) -10 (c) $\frac{1}{5}$ (d) 25

Percentages

In some contexts, and especially when making comparisons, it is conventional to express a fraction as a percentage. This is a way of expressing fractions with denominator 100; for example, $\frac{3}{5} = \frac{60}{100}$ is written as 60%.

% is read 'percent' which means 'per hundred'.

> A fraction is converted into percentage form by multiplying it by 100 and writing % after the result.

Example 1.15

Express each of the following fractions as percentages: $\frac{2}{5}$, $\frac{27}{20}$, $\frac{5}{6}$.

$\frac{2}{5} \times 100\% = \frac{200}{5}\% = 40\%$
$\frac{27}{20} \times 100\% = \frac{2700}{20}\% = 135\%$
$\frac{5}{6} \times 100\% = \frac{500}{6}\% = 83.33\%$ (to 2 d.p.)

Exercise 1.12

Express each of the following fractions as a percentage.

(a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $3\frac{3}{4}$ (d) $1\frac{1}{8}$ (e) $3\frac{3}{7}$

> To find a given percentage of a number, change the percentage to a fraction or a decimal and use it to multiply the number.

Example 1.16

Find 6% of £340.

Solution

6% is $\frac{6}{100}$ or 0.06.

So 6% of £340 is 0.06 × £340 = £20.40.

Exercise 1.13

Find each of the following.

g is the symbol for 'gram'.

(a) 40% of 250 g (b) 4% of 250 g (c) 104% of 250 g
(d) $17\frac{1}{2}$% of £150 (e) 200% of £50

> To find one quantity as a percentage of another, divide the first quantity by the second (to form a fraction) and then express the fraction as a percentage.

Example 1.17

Express 25 g as a percentage of 125 g.

Solution

$(25 \div 125) \times 100\% = \frac{25}{125} \times 100\% = 20\%.$

Exercise 1.14

(a) Express £32.50 as a percentage of £50.00

(b) Express £23 as a percentage of £115.

(c) Express £55 as a percentage of £50.

Percentage increase

Finding a new price when the original price has gone up by 10% can be done in two stages: by first finding the 10% increase and then adding it to the original price. Sometimes it is useful to find the new price using just one stage.

Example 1.18

The price of an £80 CD player goes up by 10%. What is the new price?

Solution

First method:
10% of £80 is $\frac{10}{100} \times £80 = 0.1 \times £80 = £8$, so the increase is £8.
The new price is $£80 + £8 = £88$.

Second method:
The original cost of £80 can be represented by 100%, so increasing this by 10% to 110% is the same as working out 110% of £80.

$$\frac{110}{100} \times £80 = 1.1 \times £80 = £88.$$

The advantage of this one-stage method is that it can be used to work backwards to find the original price if the new price and the percentage increase are known.

Example 1.19

The new price of a television is £225 and this is 25% more than the original price. What was the original price?

Solution

An increase from 100% to 125% needs a multiplier of $125/100 = 1.25$. So

original price $\times$ 1.25 = new price.

Taking 25% off the new price does not give the correct answer. (Try this if you are not convinced.)

So the new price divided by 1.25 gives the original price:

£225 $\div$ 1.25 = £180.

Exercise 1.15

Fill in the gaps in the table below.

Original price (£)	Percentage increase	Multiplier	New price (£)
100	30%	1.3 (130/100)	130
45	3%	1.03 (103/100)	46.35
230	15%		
120	150%		
	20%	1.2	43.20
76	17.5%	1.175	
	17.5%		30

Percentage decrease

The one-stage method can also be used for finding reduced prices.

Example 1.20

One of the £80 CD players is shop-soiled, so it is reduced by 15%. What is the new price?

Solution

The original cost of £80 can be represented by 100%, so decreasing this by 15% is the same as working out (100 – 15)% or 85% of £80. The new price is

$$\frac{85}{100} \times £80 = 0.85 \times £80 = £68.$$

Exercise 1.16

Fill in the gaps in the table below.

Original price (£)	Percentage decrease	Multiplier	New price (£)
100	20%	0.8	80
200	25%		
45	5%		
	50%		90
	30%		154

1.5 Calculating with signed numbers

Often a calculation involves a mixture of positive and negative numbers. Below is a summary of the rules which govern the arithmetic of **signed** (positive or negative) **numbers**.

> Adding a negative number is equivalent to subtracting the corresponding positive number, e.g. $4 + (-2) = 4 - 2$.
>
> Subtracting a negative number is equivalent to adding the corresponding positive number, e.g. $4 - (-2) = 4 + 2$.
>
> Multiplying or dividing two numbers with the same sign gives a positive answer, e.g. $(-4) \times (-2) = 4 \times 2$, $(-4) \div (-2) = 4 \div 2$.
>
> Multiplying or dividing two numbers with opposite signs gives a negative answer, e.g. $(+4) \times (-2) = -(4 \times 2) = -8$,
> $(+4) \div (-2) = -(4 \div 2) = -(\frac{4}{2}) = -2$.

This is often described as 'two minuses make a plus'.

Example 1.21

Evaluate each of the following.

(a) $5 + (-8)$

(b) $(-5) + (-8)$

(c) $5 - (-8)$

(d) $(-5) - (-8)$

(e) $(-7) \times (-3)$

(f) $(-7) \times 3$

(g) $(-12) \div 4$

(h) $(-12) \div (-4)$

Solution

(a) $5 + (-8) = 5 - 8 = -3$

(b) $(-5) + (-8) = (-5) - 8 = -13$

(c) $5 - (-8) = 5 + 8 = 13$

(d) $(-5) - (-8) = (-5) + 8 = 3$

(e) $(-7) \times (-3) = +(7 \times 3) = 7 \times 3 = 21$

(f) $(-7) \times 3 = -(7 \times 3) = -21$

(g) $(-12) \div 4 = -(12 \div 4) = -3$

(h) $(-12) \div (-4) = +(12 \div 4) = 3$.

Exercise 1.17

Evaluate each of the following.

(a) (i) $(-2) + (-7)$ (ii) $(-5) + 8$ (iii) $3 + (-5)$

(b) (i) $4 - (-2)$ (ii) $(-3) - (-5)$ (iii) $(-3) - (-3)$

(c) (i) $4 \times (-3)$ (ii) $(-2) \times (-7)$ (iii) $3 \times (-9)$

(d) (i) $24 \div (-6)$ (ii) $(-40) \div (-8)$ (iii) $(-45) \div 15$

1.6 Working with powers, indices and logarithms

In a number of the form a^n, a is called the **base** and n the **power** (**index** or **exponent**).

Calculations involving different bases raised to the same power can be made simpler by factorising first.

Example 1.22

Rewrite each of the following in factorised form and hence simplify it mentally.

(a) $\frac{2^2}{4^2}$ (b) $\frac{51^3}{17^3}$

Solution

(a) $\frac{2^2}{4^2} = \left(\frac{2}{4}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

(In full detail: $\frac{2^2}{4^2} = \frac{2\times 2}{4\times 4} = \left(\frac{2}{4} \times \frac{2}{4}\right) = \left(\frac{2}{4}\right)^2 = \left(\frac{2}{2\times 2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.)

(b) $\frac{51^3}{17^3} = \left(\frac{51}{17}\right)^3 = 3^3 = 27$

Exercise 1.18

Factorise and simplify each of the following mentally. Then check your result using a calculator.

(a) $\frac{100^2}{25^2}$ (b) $\frac{4^4}{2^4}$ (c) $\frac{9^3}{3^3}$

There are several rules for combining powers.

Combining powers, and negative powers

To multiply two powers with the same base, add the indices:

$a^m \times a^n = a^{m+n}$ e.g. $3^4 \times 3^2 = 3^{4+2} = 3^6$.

To divide two powers with the same base, subtract the indices:

$a^m \div a^n = a^{m-n}$ e.g. $3^4 \div 3^2 = 3^{4-2} = 3^2$.

To find the power of a power, multiply the indices:

$(a^m)^n = a^{mn}$ e.g. $(3^4)^2 = 3^{4\times 2} = 3^8$.

A number raised to a negative power is the reciprocal of the number raised to the corresponding positive power.

$a^{-m} = 1/a^m$ e.g. $3^{-2} = 1/3^2$ and $3^2 = 1/3^{-2}$.

Fractional powers

The above rules apply to fractional powers as well as integer powers. For example,

$$10^{1/2} \times 10^{1/2} = 10^{1/2+1/2} = 10^1 = 10.$$

Thus $(10^{1/2})^2 = 10$, and so $10^{1/2} = \sqrt{10}$. Similarly $10^{1/3} = \sqrt[3]{10}$.

$\sqrt[3]{10}$ is said as 'the cube root of 10'.

Example 1.23

Evaluate each of the following, first as a power, then as a number.

(a) (i) $7^2 \times 7^3$

(ii) $7^{\frac{1}{2}} \times 7^{\frac{1}{2}}$

(iii) $7^5 \div 7^3$

(iv) $7^5 \div 7^5$

(v) $7^3 \div 7^5$

Solution

(a) (i) $7^2 \times 7^3 = 7^{2+3} = 7^5 = 16\,807$

Check: $(7 \times 7) \times (7 \times 7 \times 7) = (7 \times 7 \times 7 \times 7 \times 7)$.

(ii) $7^{\frac{1}{2}} \times 7^{\frac{1}{2}} = 7^{\frac{1}{2}+\frac{1}{2}} = 7^1 = 7$

This confirms that $7^{\frac{1}{2}} = \sqrt{7}$.

(iii) $7^5 \div 7^3 = 7^{5-3} = 7^2 = 49$

Check: $\dfrac{7^5}{7^3} = \dfrac{7 \times 7 \times 7 \times 7 \times 7}{7 \times 7 \times 7} = \dfrac{7 \times 7}{1}$.

(iv) $7^5 \div 7^5 = 7^{5-5} = 7^0 = 1$

In fact, $a^0 = 1$ for any real number a.

Check: $\dfrac{7^5}{7^5} = \dfrac{7 \times 7 \times 7 \times 7 \times 7}{7 \times 7 \times 7 \times 7 \times 7} = \dfrac{1}{1} = 1$.

(v) $7^3 \div 7^5 = 7^{3-5} = 7^{-2} = \dfrac{1}{7^2} = 0.0204$ (to 4 d.p.)

Check: $\dfrac{7^3}{7^5} = \dfrac{7 \times 7 \times 7}{7 \times 7 \times 7 \times 7 \times 7} = \dfrac{1}{7 \times 7}$.

Exercise 1.19

(a) Evaluate each of the following, first as a power, then as a number.

(i) $10^2 \times 10^3$

(ii) $10^3 \times 10^{-1}$

(iii) $(-10)^3 \times (-10)^2$

(iv) $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$

(v) $10^{-2} \times 10^{-1}$

(b) Evaluate each of the following, first as a power, then as a number.

(i) $10^7 \div 10^4$

(ii) $10^4 \div 10^7$

(iii) $(-10)^7 \div (-10)^4$

(iv) $10^{\frac{3}{2}} \div 10^{\frac{1}{2}}$

(v) $10^{-2} \div 10^{-1}$

Logarithms: common and natural

Index, power and exponent all mean the same.

Before starting this section make sure you are familiar with power (index) notation and can confidently manipulate numbers written in this form.

The number 100 can be written in power notation as:

$100 = 10^2$.

$$10^2,$$

where the base is 10 and the power is 2. In this example we say that the logarithm to the base 10 of the number 100 is 2, and write

$$\log_{10} 100 = 2.$$

Logarithms were originally devised in the seventeenth century to simplify complicated calculations through the use of tables. They were also the basis of slide rules, which have been replaced by electronic calculators.

Logarithms which use the base 10 are called **common logarithms** and are written as $\log_{10}$ or sometimes simply as log.

Any positive number such as 2, 3 or 5 can be used as a base for a logarithm. **Natural logarithms** use the base e, where e is an irrational number with a value of approximately 2.71828. There are two kinds of shorthand for natural logarithms: $\log_e$ and ln. In this text we use ln.

You need to be able to change from power form to the corresponding logarithmic form and back again with confidence, 'by hand' for simple cases and using a calculator for awkward numbers.

Example 1.24

(a) Write the number 16 as a power of 4 (power form) and then write down the corresponding logarithmic form.

(b) Change $\log_5 625 = 4$ to power form and then write the power form as a simple number.

Solution

(a) Power notation: $16 = 4^2$.
Logarithmic form: $\log_4 16 = 2$. The latter is read as 'log to the base 4 of 16 equals 2'.

In both these forms, 16 is the number, 4 is the base and 2 is the index or logarithm.

(b) The base is 5 and the index is 4, so the power form is 5^4 and the simple number is 625.

Exercise 1.20

Fill in the gaps in the table below.

Power form	Logarithmic form
$81 = 3^4$	
	$\log_2 4 = 2$
$1 = 6^0$	
	$\log_5 125 = 3$
$0.2 = 5^{-1}$	
$5 = 5^1$	
	$\log_x 8 = 2$
$x = 2^{-4}$	

Here are some more examples phrased in a different way.

Example 1.25

Find the logarithm of 8 to the base 2.

Solution

Write 8 as a power of 2: $8 = 2^3$; so $\log_2 8 = 3$.

Exercise 1.21

(a) Find the logarithm to the base 2 of each of the following.

(i) 16

(ii) 2

(iii) 0.5

(b) Without using a calculator, find the logarithm to the base 10 of each of the following.

(i) 1000

(ii) 10

(iii) 0.1

You will have noticed that $\log_{10} 10$ and $\log_2 2$ have the value 1. This result is generally true, i.e. $\log_a a = 1$.

The inverse of a logarithm

If you know the logarithm of a number using a known base, how do you find the number itself? For example, if the logarithm to base 10 of some number x, say, is 4, what is x? We have

$$\log_{10} x = 4,$$

and the corresponding power form gives x:

$$x = 10^4 = 10\,000.$$

10 000 is said to be the **antilogarithm** of 4 to base 10.

Example 1.26

Find the antilogarithm of 5 to the base 2.

Solution

Here, if x is the antilogarithm,

$$5 = \log_2 x,$$

so that

$$x = 2^5 = 32.$$

So the antilogarithm of 5 to the base 2 is 32.

Exercise 1.22

(a) Without using a calculator work out the antilogarithm to the base 10 of each of the following.

(i) 5 (ii) 0 (iii) 2

(b) Find the antilogarithm to the base 3 of each of the following.

(i) 2 (ii) 1 (iii) -1

> The **logarithm** to a given base of a number is the power to which the base is raised to calculate the number.

Most of the indices encountered so far have been integers but the system of power notation extends to fractional indices. The number 1 can be written as 10^0 and the number 10 as 10^1. The numbers between 1 and 10 can also be written in power notation using the base 10 and the corresponding indices lie between 0 and 1. One of these, the index 0.5, has already been used as an alternative notation for a square root: $\sqrt{10} = 10^{0.5} = 10^{\frac{1}{2}}$.

The value of the square root of 10 is 3.162 (to 3 d.p.), that is

$$\sqrt{10} = 10^{0.5} = 3.162 \text{ (to 3 d.p.)}.$$

So the logarithm to the base 10 of $\sqrt{10} = 3.162$ (to 3 d.p.) is 0.5:

$$\log_{10} 3.162 \simeq \log_{10} \sqrt{10} = 0.5.$$

Note that, although the index 0.5 is exactly halfway between 0 and 1 the actual number (3.162) is less than halfway between 1 and 10.

Calculators may have superseded logarithms as a calculating tool but two well known measurement scales use logarithms. One is the Richter scale for earthquakes and the other is the bel (equivalent to 10 decibels) system for measuring the intensity of sound.

Logarithmic values of numbers can be found as lists in log tables but now they are more usually obtained by pressing the appropriate calculator keys. Scientific and graphics calculators have the facility to deal with both common and natural logarithms. For common logarithms, to the base 10, the key to use is generally marked [LOG] and the inverse (or antilogarithm) is the second function key marked [10^x].

For natural logarithms, to the base e, the key to use is generally marked [LN] and the inverse (or antilogarithm) is the second function key marked [e^x].

Find these keys on your calculator and establish how they work: for most scientific calculators, you enter the number then key(s); for most graphics calculators, you press the key(s) then the number and [ENTER].

Exercise 1.23

(a) Use your calculator to find each of the following, giving your answer to four decimal places.

(i) $\log_{10} 1.5489$

(ii) $\log_{10} 15.1149$

(iii) $\log_{10} 734.256$

(b) Use your calculator to find each of the following, giving your answer to four decimal places.

(i) $\ln 1.5489$

(ii) $\ln 15.1149$

(iii) $\ln 734.256$

Exercise 1.24

(a) Use your calculator to find the numbers whose logarithms to the base 10 are the following, giving your answers to four decimal places.

(i) 0.5267 (ii) 0.0023 (iii) 2.4593

(b) Use your calculator to find the numbers whose natural logarithms are the following.

(i) 0.5267 (ii) 0.0023 (iii) 2.4593

Combining logarithms

Since logarithms are indices, the rules for multiplying and dividing powers can be adapted for calculations with logarithms.

Rule 1

The log of a product is the sum of the logs. For example, for the numbers a and b:

$$\log_{10}(a \times b) = \log_{10} a + \log_{10} b,$$
$$\ln(a \times b) = \ln a + \ln b.$$

Rule 2

The log of a quotient is the difference between the logs. For example, for two numbers a and b:

$$\log_{10}\left(\tfrac{a}{b}\right) = \log_{10} a - \log_{10} b,$$
$$\ln\left(\tfrac{a}{b}\right) = \ln a - \ln b.$$

Rule 3

The log of a number raised to a power is the product of the power and the log of the number. For example, for the number a and power n:

$$\log_{10} a^n = n \log_{10} a \quad \text{and} \quad \ln a^n = n \ln a.$$

This last result will be used later for solving exponential equations (see Module 5).

Example 1.27

Simplify each of the following.

(a) $\log_{10} 6 + \log_{10} 5$ (b) $3\ln 2 - \ln 4$

Solution

(a) $\log_{10} 6 + \log_{10} 5 = \log_{10}(6 \times 5)$ (by Rule 1)

$= \log_{10} 30$

(b) $3\ln 2 - \ln 4 = \ln 2^3 - \ln 4$ (by Rule 3)

$= \ln 8 - \ln 4$

$= \ln \dfrac{8}{4}$ (by Rule 2)

$= \ln 2$

Exercise 1.25

Simplify each of the following.

(a) (i) $\log_{10} 12 + \log_{10} 3$ (ii) $\log_{10} 12 - \log_{10} 6$ (iii) $3 \log_{10} 5$

(b) (i) $\ln 15 + \ln 7$ (ii) $\ln 35 - \ln 14$ (iii) $4 \ln 3$

1.7 Working with roots

Roots are often best considered as **fractional powers**; for example, $\sqrt{2}$ written in power form is $2^{1/2}$, $\sqrt[3]{27}$ is $27^{1/3}$. However, there are some points worth understanding when using the root form.

Sometimes $\surd$ is written in place of $\sqrt{\ }$, and $\pm\sqrt{a}$ denotes both square roots of a.

- Each positive number has two **square roots**, one positive and one negative; for example, the two square roots of 9 are 3 $(3 \times 3 = 9)$ and -3 $(-3 \times -3 = 9)$. The symbol $\sqrt{\ }$ denotes the positive square root, so the two square roots of 9 are $\sqrt{9} = 3$ and $-\sqrt{9} = -3$. Calculators give only positive square roots, so you must remember about the negative ones.
- A number divided by one of its square roots gives that square root; for example,

$$\frac{3}{\sqrt{3}} = \sqrt{3} \quad \text{because} \quad \sqrt{3} \times \sqrt{3} = 3.$$

- The positive square root of a positive number can be rewritten as a product of the positive square roots of the prime factors of that number; for example, $\sqrt{6} = \sqrt{2}\sqrt{3}$. (The numbers $\sqrt{2} = 1.414\ldots$ and $\sqrt{3} = 1.732\ldots$ are irrational numbers written in **surd** form. It is convention to remove a surd in the denominator of a fraction by multiplying the numerator and denominator by that surd.)
- Sometimes it is more appropriate to simplify an expression involving square roots using factors rather than calculate a numerical solution using a calculator – particularly if the result is to be used in further calculations.

To simplify a positive square root

Rewrite the square root as the product of the positive square roots of the number's prime factors.

e.g. $\sqrt{175} = \sqrt{5 \times 5 \times 7} = \sqrt{5} \times \sqrt{5} \times \sqrt{7}$.

Multiply together pairs of identical square roots.

e.g. $\sqrt{175} = \sqrt{5} \times \sqrt{5} \times \sqrt{7} = 5\sqrt{7}$.

The perfect squares are $1(1^2)$, $4(2^2)$, $9(3^2)$, ….

The above process can be shortened if you can spot how to write a square root as the product of a perfect square and another factor. For example,

$$\begin{aligned}\sqrt{108} = \sqrt{9 \times 12} &= \sqrt{9} \times \sqrt{12} \quad \text{(9 is a perfect square)}\\ &= 3 \times \sqrt{3} \times \sqrt{4}\\ &= 3 \times \sqrt{3} \times 2 = 6\sqrt{3},\end{aligned}$$

or

$$\begin{aligned}\sqrt{108} = \sqrt{36 \times 3} &= \sqrt{36} \times \sqrt{3} \quad \text{(36 is a perfect square)}\\ &= 6\sqrt{3}.\end{aligned}$$

Example 1.28

(a) Evaluate $\sqrt{2}\sqrt{8}$ without using a calculator.

(b) Simplify $\sqrt{60}$.

Solution

(a) $\sqrt{2}\sqrt{8} = \sqrt{2 \times 8} = \sqrt{16} = 4$

(b) Write 60 as a product of its prime factors: $2 \times 2 \times 3 \times 5$. Then

$$\sqrt{60} = \sqrt{2 \times 2 \times 3 \times 5} = \sqrt{2} \times \sqrt{2} \times \sqrt{3} \times \sqrt{5} = 2\sqrt{3}\sqrt{5} = 2\sqrt{15}.$$

Exercise 1.26

(a) Evaluate each of the following without a calculator.

(i) $\sqrt{100}$ (ii) $\dfrac{9}{\sqrt{9}}$ (iii) $\dfrac{\sqrt{18}}{\sqrt{2}}$

(b) Simplify each of the following by factorising first.

(i) $\sqrt{200}$ (ii) $\sqrt{112}$ (iii) $\sqrt{256}$ (iv) $\dfrac{\sqrt{15}}{\sqrt{3}}$

Exercise 1.27

(a) A square has edge length a. What is the length of its diagonal?

(b) The longest edge of an isosceles right-angled triangle is a. What is the length of the shorter edges?

A-size paper has edges of lengths 1 and $\sqrt{2}$.

(c) What is the length of the diagonal of A-size paper?

(d) Show that cutting A-size paper in half, parallel to the shorter edge, creates two rectangles that are similar to the original rectangle.

(e) The largest possible square is cut from an A-size rectangle. What are the dimensions of the waste piece?

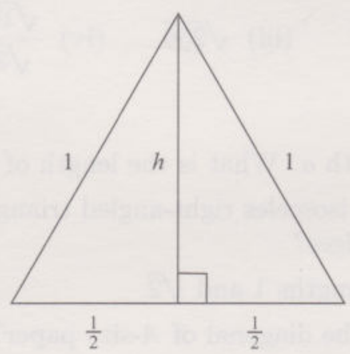
(f) Two equal squares (as large as possible), side by side, are cut from A-size paper, leaving a rectangle. Write down the dimension and the area of each piece. Check your answer against the area of the original rectangle.

Surd form

In some of the exercises above we have left an answer in *surd form* (e.g $\sqrt{2}$ and $\sqrt{5}$) where there was no *exact* numerical form for the number. If you input $\sqrt{2}$ into your calculator it will display something like 1.414 213 562. This is, of course, a very good approximation to $\sqrt{2}$, more accurate than we should need for most purposes, but it is an *approximation*. In fact, there is no way to write down the exact value of $\sqrt{2}$ except by using this symbol. So, in many circumstances, rather than write down some approximation – like 1.4 or 1.4142 – we leave the result in surd form as $\sqrt{2}$. This ensures that our answer is absolutely correct, and enables us to ensure that subsequent calculations are accurate. For example, suppose that the exact answer to a question is $\sqrt{2}$, and we write it down as $\sqrt{2}$ rather than 1.4 or 1.4142. If, later, we need to square this answer then it will be calculated as $\sqrt{2} \times \sqrt{2} = 2$ rather than $1.4 \times 1.4 = 1.96$ or $1.4142 \times 1.4142 = 1.999\,961\,64$.

You can use Pythagoras' Theorem to calculate the height of an equilateral triangle. For example, for the triangle shown in the figure,

$$h^2 + (\tfrac{1}{2})^2 = 1^2 = 1.$$



So, if we have $h^2 = \frac{3}{4}$, then $h = \sqrt{3/4} = \sqrt{3}/2$. Frequently we write this as $\frac{\sqrt{3}}{2}$. Here $\sqrt{3}$ is a surd, standing for the exact positive square root of 3. To three decimal places, $\sqrt{3} = 1.732$, but
$(1.732/2)^2 + (\frac{1}{2})^2 = (0.866)^2 + 0.25 = 0.999\,56$, while
$(\sqrt{3}/2)^2 + (\frac{1}{2})^2 = \frac{3}{4} + \frac{1}{4} = 1$ exactly.

Surds can be manipulated exactly as you would manipulate other numbers. Some examples are given below.

Example 1.29

Without using a calculator, simplify the following.

(a) $\sqrt{2} \times \sqrt{6}$ (b) $4/\sqrt{2}$ (c) $\sqrt{18}$

Solution

(a) $\sqrt{2} \times \sqrt{6} = \sqrt{2} \times \sqrt{2} \times \sqrt{3} = 2 \times \sqrt{3}$ (Usually written more simply as $2\sqrt{3}$.)

(b) $4/\sqrt{2} = 2 \times 2/\sqrt{2} = 2 \times \sqrt{2} \times \sqrt{2}/\sqrt{2} = 2\sqrt{2}$

(c) $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

Exercise 1.28

Without using a calculator, simplify the following.

(a) $\sqrt{2} \times \sqrt{24}$ (b) $\sqrt{18}/3$ (c) $12/\sqrt{3}$

1.8 Ratio and proportion

Ratio

Ratio is a fundamental theme running throughout mathematics; it is one of the ways of comparing one thing with another and is used extensively in algebra, trigonometry and geometry. The concept of ratio is also used in converting from one unit to another using a **scale factor**, and in constructing pie charts, in comparing prices and earnings. It is essential to understand the concept and be able to manipulate ratios confidently.

Comparisons can be made in different ways. Take the example of two people, Alan and Beth, who contribute to a joint budget. Alan puts in £100 and Beth £200, making the total budget £300. One way of comparing their individual contributions is to say that Beth contributes £100 more than Alan; this makes use of the absolute (or actual) difference between the two amounts.

Another way is to look at the relative amounts. This can be done by comparing *part* with *part*, i.e. comparing Beth's part with Alan's.

(i) Beth's contribution as compared to Alan's is in the ratio 200 to 100. This ratio is written $200:100$ and simplifies to $2:1$. This simplified form of the ratio $200:100$ is obtained by dividing each quantity by their HCF. $2:1$ is the expression of this ratio 'in its lowest terms'.

(Looking at it the other way, round Alan's contribution compared with Beth's is in the ratio $1:2$.)

(ii) Beth contributes twice as much as Alan. This uses a scale or multiplicative factor of 2 obtained by dividing Beth's contribution by Alan's. £200 divided by £100 is $\frac{200}{100} = 2$.

(Looking at it the other way round, Alan contributes half as much as Beth: £100/£200 $= \frac{1}{2}$.)

In both these ways, the result is a number which has no units.

The concept of ratio is closely connected with fractions but in the case of fractions each *part* is being compared with the *whole*. In the above situation the total or whole budget is the sum of the individual contributions, i.e. £300.

Beth's contribution is £200 out of £300, which is $\frac{200}{300} = \frac{2}{3}$ of the total budget.

Alan's contribution is £100 out of £300, which is $\frac{100}{300} = \frac{1}{3}$ of the total budget.

Here, the answers $\frac{2}{3}$ and $\frac{1}{3}$ only have any meaning if they are linked with a quantity, in this case 'the total budget'.

Example 1.30

Two people contribute to a budget in the ratio 10 to 30. Put this ratio in its lowest terms and write each contribution as a fraction of the whole budget.

Solution

Divide each quantity in the ratio $10:30$ by their HCF, in this case 10.

The ratio becomes $1:3$.

There are 4 parts altogether, so one person contributes $\frac{1}{4}$ of the whole budget and the other contributes $\frac{3}{4}$.

Exercise 1.29

Complete the table.

Ratio (comparing part with part)	Ratio (in lowest terms)	As fractions (parts of the whole)
$4:16$		
$10:5$		
$6:9$		
$1:7$		
$10:20:30$		
$30:25$		

Example 1.31

£600 is to be shared between 2 people in the ratio $2:3$. How much will each person get?

Solution

There are 5 parts, so one person will get $\frac{2}{5}$ of £600, which is £240. The other person will get $\frac{3}{5}$ of £600, which is £360.

Exercise 1.30

Share each of the following amounts of money in the ratio shown.

(a) £400 between 2 people in the ratio $3:5$

(b) £250 between 3 people in the ratio $1:2:2$

(c) £1000 between 2 people in the ratio $3:2$

Map scales

Maps and plans are drawn to scale to represent something in the physical world. In a scale drawing, all lengths are multiplied by the same **scale factor** often denoted by the letter k.

Example 1.32

A map has a scale of $1:25\,000$. This means that the scale factor is $25\,000$.

(a) A footpath alongside a canal measures 12.5 centimetres on the map. How far, in kilometres, would it be to walk along this path?

(b) The distance between two places by road is 8 km. What length would this be represented by on the map?

Solution

(a) 12.5 cm on the map represents a distance of $12.5 \times 25\,000$ cm.
$12.5 \times 25\,000\text{ cm} = 312\,500\text{ cm}$.

There are 100 cm in a metre, so the length of the path is

$$\frac{312\,500}{100}\text{ cm} = 3125\text{ m or } 3.125\text{ km}.$$

(b) $$\begin{aligned} 8\text{ km} &= 8 \times 1000\text{ m} \\ &= 8 \times 1000 \times 100\text{ cm} \\ &= 800\,000\text{ cm} \end{aligned}$$

On the map this length would be represented by $\frac{800\,000}{25\,000}$ cm $= 32$ cm.

Exercise 1.31

A map has a scale $1:50\,000$.

(a) What is the actual length of a road shown as 16.5 cm?

(b) What would be the length on the map of a path of length 15 km?

Sometimes ratios are given as single numbers; for example, the ratio of the circumference of a circle to its diameter is a fixed number called **pi**, the symbol for which is π.

$\pi = 3.141\,59\ldots$

Enlargement

In the enlargement of a photograph, the photo has to stay 'in proportion' in order to avoid distortion. That is, the proportionate increase in height of the photo is the same as the proportionate increase in width.

Example 1.33

A photo has a height of 15 cm and a width 12 cm. What will be the new width if the height is increased to 20 cm?

Solution

The length multiplier is $\frac{20}{15}$. The width will go up in the same proportion, so the same multiplier is used. The new width is

$$\frac{20}{15} \times 12\text{ cm} = 16\text{ cm}.$$

Exercise 1.32

Complete the table showing the different enlargement or reduction sizes for the photograph above.

Height	15	20	30		10	
Width	12			6		40

Proportion

Alan and Beth decide that they need to put more money into their budget but they agree to keep the amounts in the same proportion, i.e. Beth will still pay twice as much as Alan. If Alan contributes £110, then Beth will put in £220. The ratio is $220 : 110 = 2 : 1$, i.e. it stays the same.

Exercise 1.33

In the above situation, if Beth pays £450, how much will Alan contribute?

Direct and inverse proportion

If two quantities are **directly proportional** to each other, multiplying one quantity by a number means that the other quantity is multiplied by the same number. An example of direct proportion is: the greater your running speed, the greater the distance travelled in a fixed time. We say that speed is directly proportional to distance travelled.

If two quantities are **inversely proportional**, multiplying one quantity by a number means that the other quantity is divided by the same number (or multiplied by its reciprocal). An example of inverse proportion is: the faster the speed of a journey, the shorter the time taken. In this case, speed is inversely proportional to time taken.

We shall return to these ideas in Module 7.

Module 2 Measures

Measures such as a count of 10 000 cars and a speed of 10 km per hour illustrate the need for quantities to be labelled and for units to be specified, so that they can be compared and used to establish relationships.

2.1 Units

Système International d'Unités (The international system of units)

This internationally agreed system, referred to as SI units, is based on the metric system. It comprises a limited number of **base units** which have associated symbols. Examples are metre (m) for length, second (s) for time, litre (l) for capacity. For large or small quantities these base units are combined with prefixes to indicate magnitude in powers of ten; for example, kilometre (km) and millisecond (ms). The table shows some of these prefixes.

Table 2.1

Prefix	Symbol	Figures	Words	Powers of 10
mega	M	1 000 000	million	10^6
kilo	k	1 000	thousand	10^3
		1	one	10^0
centi	c	0.01	hundredth	10^{-2}
milli	m	0.001	thousandth	10^{-3}
micro	μ	0.000 001	millionth	10^{-6}

μ (mu) is a Greek letter, pronounced 'mew'. The full Greek alphabet is given in the *Guide to Preparation.*

Base units can be combined to produce **derived units**. Here are some examples.

area	m^2	(square metres)
volume	m^3	(cubic metres)
velocity	$\mathrm{m\,s^{-1}}$	(metres per second)
acceleration	$\mathrm{m\,s^{-2}}$	(metres per second per second or metres per second squared)

Note that in a derived unit, a fine space is placed between base units.

2.2 Inequalities

The = sign indicates that two 'things' have exactly the same value. In everyday life you are likely to meet situations which involve a range of values and use phrases such as 'more than' or 'less than' or their equivalent in expressions like 'all prices are less than £1' and 'the population of Milton Keynes is greater than 150 000'.

These expressions are examples of **inequalities**. The mathematical symbols used to express relationships of inequality are as follows.

Symbol	**Meaning**
$<$	is less than
$\leq$	is less than or equal to
$>$	is greater than
$\geq$	is greater than or equal to

Remember: if you think of an inequality symbol as an arrow it points to the smaller value, as Figure 2.1 shows.

is greater than	is less than
larger > smaller	smaller < larger

Figure 2.1

Examples of true statements involving the inequality symbols $<$ and $>$ are

$5 < 7,\ 7 > 5,\ -1 < 0,\ -5 > -7.$

True statements, involving $\leq$ and $\geq$ are

$1 \leq 1$ and $1 \geq 1$.

These symbols can also be used with unknowns. For example, if P is the population of Milton Keynes then the second introductory example may be written

$P > 150\,000.$

The pictorial representation of inequalities such as $P > 150\,000$ are looked at in Module 5.

Exercise 2.1

(a) Write each of the following statements in words.

(i) $16 > 12$

(ii) $-11 < -9$

(b) Write each of the following statements in symbols.

(i) -9 is greater than -11.

(ii) The number of students in the classroom is at least 20.

2.3 Formulas

An expression of the relationship between quantities as an equality is called a **formula**; for example:

circumference of a circle = 2π times radius.

Formulas are often expressed in symbols; for example:

$$C = 2\pi r,$$

where C is the circumference and r is the radius.

Another example is:

area of a triangle $= \frac{1}{2}$ base $\times$ height

or, using the obvious symbols b for base and h for height (see Figure 2.2),

$$A = \tfrac{1}{2}bh.$$

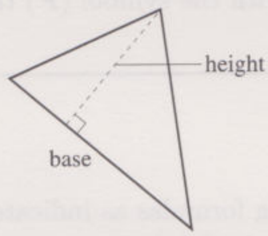


Figure 2.2

The two formulas above involve symbols which include units. For example, if $r = 6\,\text{cm}$ then

$$C = 2\pi \times 6\,\text{cm} = 12\pi\,\text{cm}.$$

Sometimes the symbols in a formula represent just numerical qualities. An example is as follows.

To convert from degrees Fahrenheit to degrees Celsius, take the Fahrenheit measurement and subtract 32; multiply the result by $\frac{5}{9}$. Using the obvious symbols for the numbers involved gives the formula

$$C = \tfrac{5}{9}(F - 32).$$

Formulas are in effect a recipe for doing a calculation when all but one of the quantities is known. Substituting in formulas is straightforward provided consideration is given to the units, where appropriate, and the formula is 'the right way' round. Sometimes a formula needs to be rearranged or 'undone'. One way of doing this is illustrated below.

If you are confident in an alternative method, use that.

Example 2.1

'Undo' the formula $C = \frac{5}{9}(F - 32)$ so that it reads 'F =something'.

Solution

We represent the formula $C = \frac{5}{9}(F - 32)$ as a diagram to show the order of the operations, from left to right.

$$F \xrightarrow{-32} F - 32 \xrightarrow{\times\frac{5}{9}} \tfrac{5}{9}(F - 32)(= C) \tag{2.1}$$

To undo the formula to obtain

$F =$ something,

we undo each of the above operations in reverse order, i.e. starting at the right, at C.

$$(F =)\tfrac{9}{5}C + 32 \xleftarrow{+32} \tfrac{9}{5}C \xleftarrow{\div\frac{5}{9}} C$$

This gives the formula

$$F = \tfrac{9}{5}C + 32.$$

The important feature to notice in producing these diagrams is that you need to start (Equation 2.1) with the symbol (F) that you wish to make the 'subject' of the formula.

Exercise 2.2

Rearrange each of the following formulas as indicated. You should use the 'doing and undoing' method in part (a).

(a) The area of a triangle formula, $A = \frac{1}{2}bh$, to give a formula for h, the height, in terms of A and b.

(b) speed $= \dfrac{\text{distance}}{\text{time}}$ to give a formula for

(i) distance, (ii) time.

(c) The circumference of a circle formula, $C = 2\pi r$, to give a formula for r, the radius.

2.4 Measuring and classifying angles

An **angle** is an amount of rotation or turning, normally taken as positive when measured in the anticlockwise sense – angles measured clockwise are taken to be negative; see Figure 2.3.

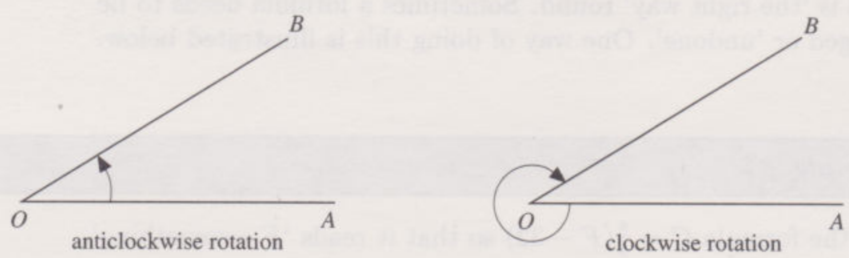


Figure 2.3

Angles can be measured in various units – for example, degrees, radians and gradients, but only the first two are usually used in mathematics.

The Babylonians divided the circumference of a circle into 12 equal sections and divided each of those sections into 30 equal parts. The resulting 'part' corresponds to the present-day **degree**, because there are 360 degrees in a full revolution. Figure 2.4 shows how the Babylonian and the present-day systems are related.

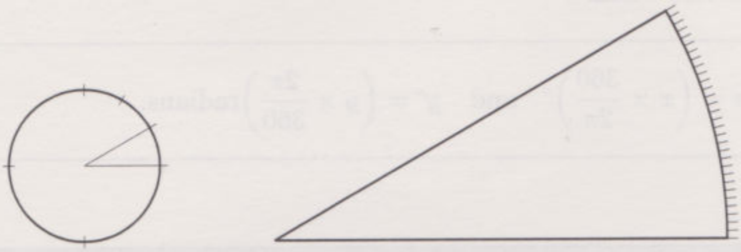


Figure 2.4

A less arbitrary way of dividing the circumference of a circle ($2\pi r$) is to divide it by the radius (r). This gives $\frac{2\pi r}{r} = 2\pi$ parts for a complete revolution, for any radius.

It is this observation which leads to the definition of a **radian**.

The angle subtended at the centre of a circle by an arc equal in length to the radius is defined to be one **radian**; see Figure 2.5. There are 2π radians in a full revolution.

An **arc** of a circle is any part of the circle joining two points on the circle.

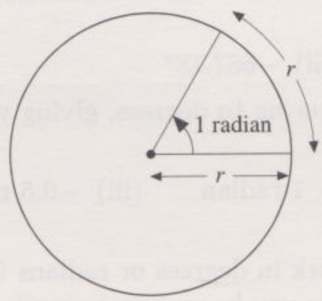


Figure 2.5

Exercise 2.3

Draw rough diagrams to indicate each of the following angles. Label the angles in both degrees and radians. (The symbol for the degree is °; for example, 360 degrees is written 360°.)

(a) $\frac{1}{4}$ revolution

(b) $\frac{1}{3}$ revolution

(c) $-\frac{1}{6}$ revolution

Converting angle measures

Since 2π radians $= 360°$,

$$1 \text{ radian } = \frac{360°}{2\pi} \text{ and } 1° = \frac{2\pi}{360} \text{ radians.}$$

1 radian $= 57.30°$ (to 2 d.p.);
$1° = 0.017$ radians (to 3 d.p.).

These equivalences lead to the rules below for conversion from radians to degrees, and vice versa.

$$x \text{ radians} = \left(x \times \frac{360}{2\pi}\right)^\circ \quad \text{and} \quad y° = \left(y \times \frac{2\pi}{360}\right)\text{radians.}$$

Example 2.2

Convert 113.5° to radians.

Solution

$$113.5° = \left(113.5 \times \frac{2\pi}{360}\right)\text{radians}$$

$$= 1.98 \text{ radians (to 2 d.p.)}$$

Exercise 2.4

Unless degrees are specified, you should assume an angle measurement is given in radians.

(a) Convert each of the following to radians, giving your answer to two decimal places.

(i) 54° (ii) 125° (iii) −667.18°

(b) Convert each of the following to degrees, giving your answers to one decimal place.

(i) $2\pi/7$ radians (ii) 1 radian (iii) −0.5 radians

Calculators can be set to work in degrees or radians (and sometimes gradients as well). Make sure you know how to work in degrees and radians on your calculator.

Some calculators can be used to convert between degrees and radians automatically. If this function is available on your calculator, repeat the above exercise using it.

Classifying angles

Several different types of (positive) angle are named, as follows.

An **acute angle** is less than a quarter revolution – that is, less than 90°, or $\pi/2$ radians.

acute angle

Figure 2.6

A **right angle** is a quarter revolution – that is, exactly 90°, or $\pi/2$ radians.

right angle
90°
quarter revolution

Figure 2.7

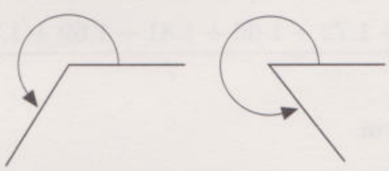
Note the use of the mark to denote a right angle.

An **obtuse angle** is between 90° and 180° (but not equal to either) – that is, between $\pi/2$ and π radians.

obtuse angle

Figure 2.8

A **reflex angle** is between 180° and 360° – that is, between π and 2π radians.



reflex angles

Figure 2.9

2.5 Statistical measures

Statistics involves collecting, analysing, interpreting and communicating **numerical data**. One aspect of analysing a set of data is to compare it with another set, so there needs to be agreement about forms of measure. One way of providing a summary of a batch of data is an average, and there are three types of average: **mean, mode** and **median**. Which is most appropriate to use depends on what the average is to be used for; in some cases more than one may be needed.

Mean

The **mean** is what most people and the media use when talking about an average. It is found by dividing the sum of all the item values in a batch by the number of items (the **batch size**). This gives the formula

$$\text{mean} = \frac{\text{sum of values}}{\text{batch size}}.$$

The mathematical symbol for 'sum of' is the Greek letter capital sigma Σ. If the values are denoted by x, the batch size by n, and the mean by

$\bar{x}$ (read as x bar),

then the formula for the mean may be written in symbols as

$$\bar{x} = \frac{\Sigma x}{n} \text{ (read as } x \text{ bar equals sigma } x \text{ over } n).$$

Some scientific calculators have statistical functions which use these symbols to denote the keys to use. The values are entered by using a special key; on some calculators this is marked [Σ+].

Example 2.3

The heights in metres (measured to the nearest centimetre) of a group of seven people are given below.

1.52 1.72 1.66 1.81 1.69 1.59 1.77

What is the mean height of the group?

Solution

The number of values in the batch (the batch size) is 7. Hence

$$\begin{aligned}\text{mean height} &= \frac{1.52 + 1.72 + 1.66 + 1.81 + 1.69 + 1.59 + 1.77}{7}\text{ m}\\ &= \frac{11.76}{7}\text{ m}\\ &= 1.68\text{ m}.\end{aligned}$$

Exercise 2.5

Find the mean of each of the following batches of data.

(a) 23 21 26 26 24

(b) 101 107 98 92 115 102

Data are usually listed differently when several values occur more than once and care needs to be taken to include all the data. In these cases, it is more appropriate to use an adapted formula for the mean, as shown in the following example.

Example 2.4

A number of people were asked what size of household they lived in – that is, how many people lived in their household. They gave the responses shown in Table 2.2. What is the mean number of people per household?

Table 2.2

Size of household	Number of responses
1	3
2	2
3	1
4	3
5	1

Solution

Let the values of size of household be denoted by x, and the number of each such household by f (for frequency). The formula for the mean may then be written as

$$\frac{\sum xf}{\sum f},$$

where $\sum$ means 'sum of'. The calculation is as follows.

$$\begin{aligned}\text{Total number of people in the households} &= (1 \times 3) + (2 \times 2) + (3 \times 1) \\ &\quad + (4 \times 3) + (5 \times 1) \\ &= 27 \text{ (this is } \textstyle\sum xf).\end{aligned}$$

$$\begin{aligned}\text{Total number of households} &= 3 + 2 + 1 + 3 + 1 \\ &= 10 \text{ (this is } \textstyle\sum f).\end{aligned}$$

$$\text{Mean number of people per household} = \frac{\sum xf}{\sum f} = \frac{27}{10} = 2.7.$$

Exercise 2.6

Table 2.3 below gives information on the number of brothers and sisters that children have in a particular school class. Find the mean number of siblings of the children, correct to one decimal place.

Table 2.3

Number of siblings	Frequency
0	7
1	18
2	5
3	2
4	0
5	1

Limitations!

One limitation of means is that extreme values can cause distortions and make the summary meaningless. (For example, the mean of 101, 1, 1, 1 and 1 is 21.) To make a mean a more informative summary, the **range** (the difference between the highest and lowest values) should also be quoted. (In the above example, the range is 100.)

Mode

The **mode** of a batch of data is the most frequently occurring value. For some batches of data and for some purposes this might be the most representative. In the data in Table 2.3, the mode is 1 sibling (with frequency 18).

Median

The **median** is essentially the middle value – that is, the middle value when the values are placed in size order – of a batch of data.

It is found by the following procedure.

Any repeat values must be listed.

- ◇ First sort the values in the batch into ascending order.
- ◇ If the batch size is odd, then the median is the middle value in the list.
- ◇ If the batch size is even, then the median is the mean of the two middle values.

Example 2.5

(a) What is the median height for the data in Example 2.3?

(b) Suppose that the tallest person is removed from the data in Example 2.3, leaving the following six values.

1.52 1.59 1.66 1.69 1.72 1.77

What is the median height of the modified batch?

Solution

(a) Sorting the seven heights in Example 2.3 into ascending order produces the following.

1.52 1.59 1.66 1.69 1.72 1.77 1.81

Since the batch size is odd, the median is the middle value, which is 1.69 m.

(b) There are now two middle values, namely, 1.66 and 1.69. Thus

$$\begin{aligned}\text{median} &= \frac{1.66 + 1.69}{2}\ \text{m}\\ &= 1.675\ \text{m}.\end{aligned}$$

Note: In this answer, more decimal places are used than in the original data.

Exercise 2.7

Find the median of each of the following batches of data.

(a) 23 21 26 26 24

(b) 74 38 57 93 47 64 71

(c) 7 7 9 6 4 10 8 12

(d) 101 107 98 92 115 102

Module 3 Some basic figures

3.1 Triangles

Angle sum of a triangle

A **triangle** is a plane closed figure bounded by three (straight) line segments. These line segments, which are called the **sides** of the triangle, form three angles which add up to a fixed sum. By convention the lengths of the sides of a triangle are labelled with lower-case letters each corresponding to the upper-case letter of the opposite vertex, as shown below.

A 'plane figure' is one which is located entirely in a plane.

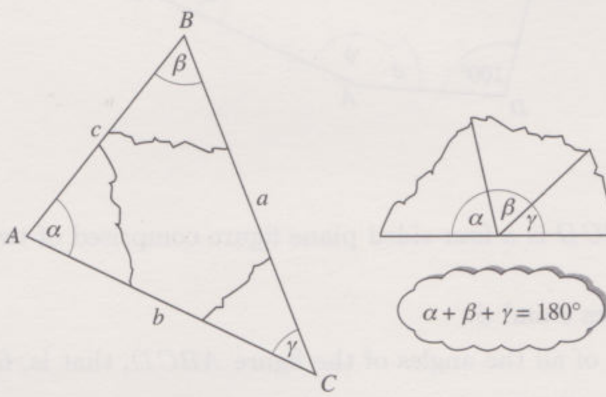


Figure 3.1

For any triangle ABC, the sum of its angles is 180° (or π radians).

Greek letters such as α (alpha), β (beta), γ (gamma), are frequently used for angles. The full Greek alphabet is given in the *Guide to Preparation*.

Example 3.1

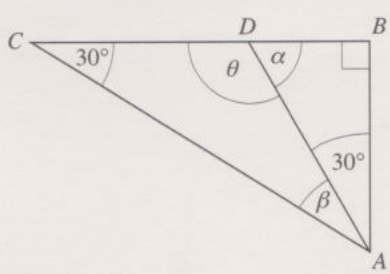


Figure 3.2

Find the angles α, β and θ in Figure 3.2.

The mark at B indicates that the angle at B is 90°.

ΔABC is shorthand for 'the triangle ABC'.
$\angle ABD$ is shorthand for 'the angle ABD', i.e. the angle between the lines AB and BD.

Solution

In ΔABD, $\angle ABD = 90°$ and $\angle BAD = 30°$.

Thus $\alpha + 30° + 90° = 180°$, and so $\alpha = 60°$.

CDB is a straight line.

Thus $\theta + \alpha = \angle CDB = 180°$, and so $\theta = 180° - \alpha = 120°$.

In ΔADC, $\angle ACD = 30°$ and $\angle CDA = \theta = 120°$.

Thus $\beta + 30° + 120° = 180°$, and so $\beta = 30°$.

Exercise 3.1

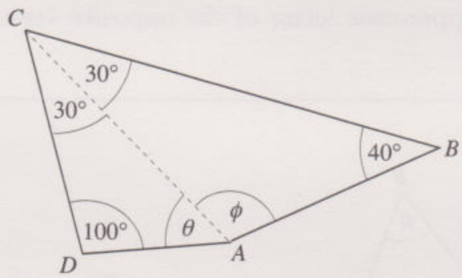


Figure 3.3

In Figure 3.3, $ABCD$ is a four-sided plane figure comprised of two triangles.

(a) Find the angles θ and ϕ.

(b) Find the sum of all the angles of the figure $ABCD$, that is, find

$$\angle ABC + \angle BCD + \angle CDA + \angle DAB.$$

Classifying triangles

Several different types of triangle are named, as follows.

A **right-angled triangle** has one angle which is a right angle. The other two angles must add up to 90°.

Figure 3.4

An **equilateral** triangle has all sides of equal length. Its angles are therefore also all equal; each is 60°.

In these figures, note how equal angles and sides of equal or unequal length are indicated.

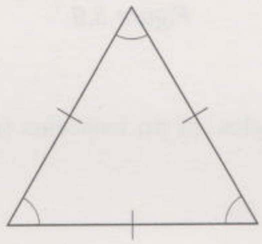


Figure 3.5

An **isosceles** triangle has two sides of equal length. Two 'base' angles are equal and acute. The third angle can be acute, obtuse or a right angle.

All equilateral triangles are also isosceles.

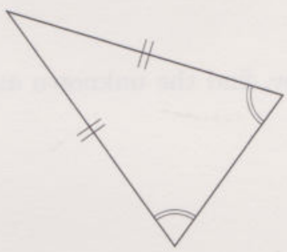


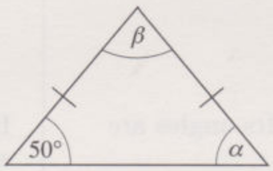
Figure 3.6

A **scalene** triangle has all three sides of different lengths. Its angles are also all different, but at least two are acute, while the third can be acute, obtuse or a right angle.

Figure 3.7

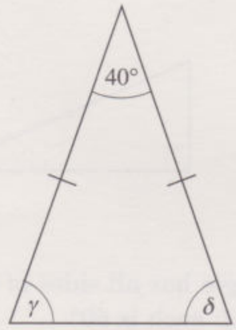
Example 3.2

Find the unknown angles in each of the following isosceles triangles.



(a)

Figure 3.8



(b)

Figure 3.9

Solution

(a) α and $50°$ are the 'base angles' of an isosceles triangle, so $\alpha = 50°$. Hence

$$\beta = 180° - 50° - 50° = 80°.$$

(b) γ and δ are the 'base angles' here, so $\gamma = \delta$. By the angle sum property of triangles,

$$40° + \gamma + \delta = 40° + 2\gamma = 180°.$$

Therefore $2\gamma = 140°$, and hence $\gamma = \delta = 70°$.

Exercise 3.2

For each of the triangles below, find the unknown angle(s) and state what kind of triangle it is.

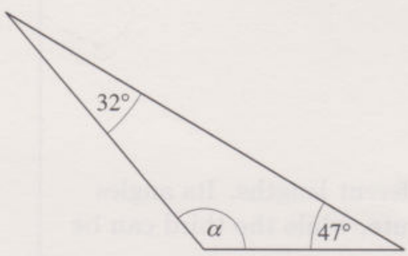


Figure 3.10

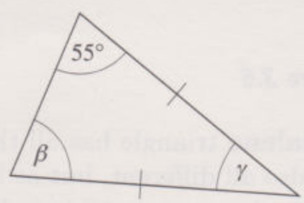


Figure 3.11

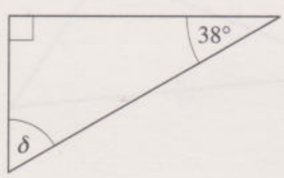


Figure 3.12

Pythagoras' Theorem

If two sides of a right-angled triangle are known, then the third side can be calculated using **Pythagoras' Theorem**. In a right-angled triangle the side opposite the right angle is called the **hypotenuse**.

The hypotenuse is the longest side of a right-angled triangle.

> Pythagoras' Theorem states that in any right-angled triangle the square (of the length) of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

In Figure 3.13, this means that if a, b and c represent the lengths of the sides of the triangle, then $c^2 = a^2 + b^2$. In terms of areas, as shown on the right, the area of square 1 equals the sum of the areas of squares 2 and 3.

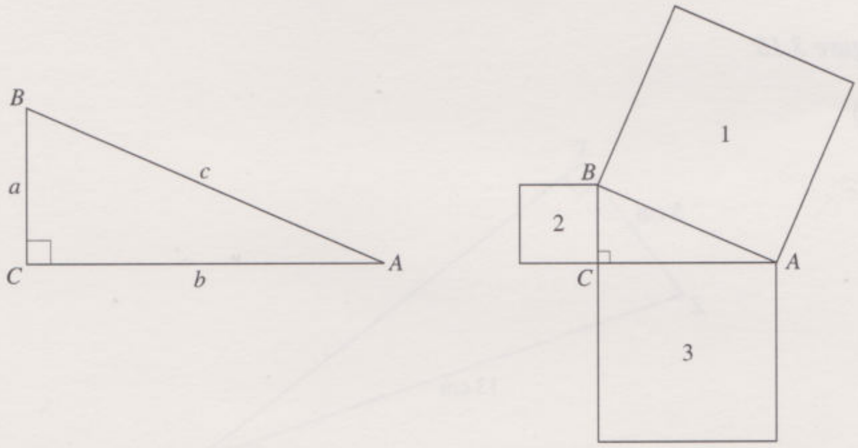
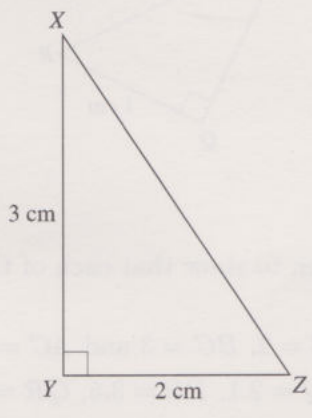


Figure 3.13

Example 3.3

Find the length of the hypotenuse in Figure 3.14.



In this example, XZ, XY and YZ denote lengths and sides.

This example also illustrates the common practice of omitting units where they are obvious and cumbersome. The appropriate unit is reinstated at the end.

Figure 3.14

Solution

Side XZ is the hypotenuse. By Pythagoras' Theorem

$$XZ^2 = XY^2 + YZ^2,$$

so, omitting the units, $XZ^2 = 3^2 + 2^2 = 9 + 4 = 13$. Hence
$XZ = \sqrt{13} = 3.61\,\text{cm}$ (to 2 d.p.).

Exercise 3.3

(a) Find the missing length in each of the following triangles.

(i)

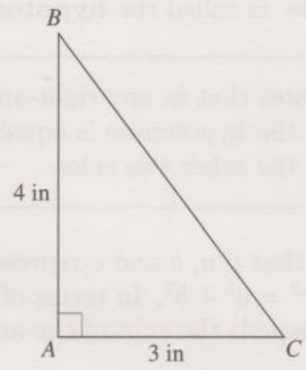


Figure 3.15

(ii)

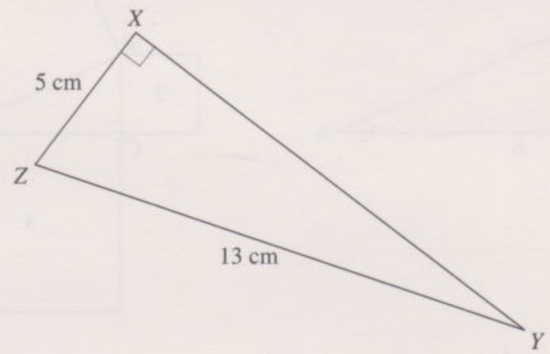


Figure 3.16

(iii)

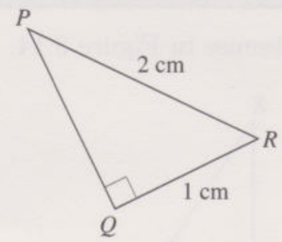


Figure 3.17

(b) Use Pythagoras' Theorem to show that each of the following triangles is not right-angled.

(i) ΔABC in which $AB = 2$, $BC = 3$ and $AC = 4$

(ii) ΔPQR in which $PQ = 2.1$, $PR = 3.5$, $QR = 4$

3.2 Rectangles

Angle sum of a quadrilateral

A **quadrilateral** is a plane, closed figure with four bounding sides. Its four angles add up to a fixed sum which is twice the angle sum of a triangle (see Exercise 3.1 (b)).

> The sum of the angles of a quadrilateral is $360°$ (or 2π radians).

> A quadrilateral with all angles equal (and hence right angles) is called a **rectangle**. Its diagonals bisect each other and its opposite sides are equal in length.

Notice that a square is a special case of a rectangle.

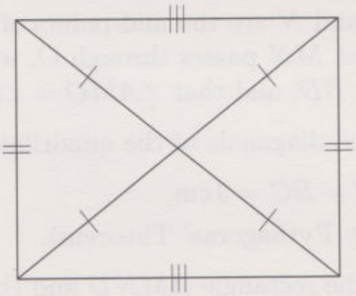


Figure 3.18

Perimeters

The lengths of the equal sides of a rectangle are often denoted by l (length) and b (breadth).

- ◇ The perimeter of a rectangle is

  $l + b + l + b = 2l + 2b = 2(l + b)$.

- ◇ The perimeter of a triangle with sides of length a, b and c is

  $a + b + c$.

Example 3.4

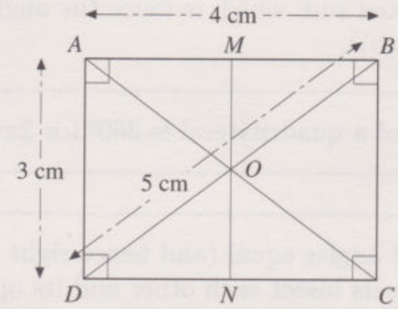


Figure 3.19

$ABCD$ is a rectangle. M and N are the mid-points of AB and DC, respectively. (It follows that MN passes through O, which is the mid-point of both AC and BD, and that $\angle AMO = \angle DNO = 90°$.)

The lengths of the sides and diagonals of the quadrilateral are as follows:

$AB = DC = 4\,\text{cm}, AD = BC = 3\,\text{cm},$

$AC = BD = 5\,\text{cm}$ (by Pythagoras' Theorem).

Find the perimeter of (a) the rectangle $AMND$ and (b) the triangle AOD.

Solution

(a) In the rectangle $AMND$, $AD = 3\,\text{cm}$, $AM = \frac{4}{2} = 2\,\text{cm}$. So the perimeter of $AMND$ is $2(3 + 2) = 2 \times 5 = 10\,\text{cm}$.

(b) In the triangle AOD, $AD = 3\,\text{cm}$, $AO = OD = \frac{5}{2}\,\text{cm}$. So the perimeter of AOD is $3 + \frac{5}{2} + \frac{5}{2} = 8\,\text{cm}$.

Exercise 3.4

Find the perimeters of the following figures in the diagram in Example 3.4.

(a) rectangle $ABCD$

(b) triangle DBC

(c) triangle DOC

(d) triangle DON

3.3 Circles

A **circle** can be defined as the set of all points in a plane which are a fixed distance from a fixed point in that plane. This fixed point is called the **centre** of the circle and is often indicated with the letter O.

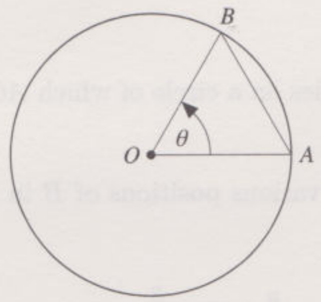


Figure 3.20

The fixed distance is called the **radius** of the circle (as is any straight line segment from O to the circle) and the distance round the circle is called the **circumference**.

The plural of radius is radii.

Two points A and B on a circle divide the circle into two parts or **arcs**. Consider the smaller of the two arcs, which is referred to as arc AB. The region bounded by the arc AB and the radii OA and OB is called the **sector** OAB. The angle AOB ($\angle AOB = \theta$) is called the angle **subtended** by the arc AB. The line segment between A and B is a **chord**.

The two arcs are equal in length only if the line segment AB is a diameter.

There is also a sector and a subtended angle associated with the larger arc AB. (Depending on the context, arc AB denotes either the smaller or the larger arc.)

Any straight line segment joining two points on a circle and which passes through the centre of the circle is called a **diameter**, so the length of a diameter is twice that of the radius. The word 'diameter' is also used to denote the length of a diameter.

The circumference, C, of a circle is a multiple of its diameter, and found using either of the formulas below.

$$C = \pi d \quad \text{or} \quad C = 2\pi r,$$

where d is the diameter of the circle and r is its radius.

π is the ratio of the circumference of a circle to its diameter.

Example 3.5

(a) Find the circumference of a circle of radius 14 cm, correct to two significant figures.

(b) Find the diameter of a circle of circumference 35 m, correct to three decimal places.

Solution

(a) $C = 2\pi r = 2 \times \pi \times 14 = 87.964\,594\ldots = 88\,\text{cm}$ (to 2 s.f.)

(b) $C = \pi d = 35$, so $d = \dfrac{35}{\pi} = 11.140\,846\ldots = 11.141\,\text{m}$ (to 3 d.p.)

Exercise 3.5

(a) Find the circumference of a circle of radius 7 cm, correct to two decimal places.

(b) Find the diameter of a circle of circumference 44 m, correct to three decimal places.

Angle in a semicircle

Let ΔABC be such that B lies on a circle of which AC is a diameter. Then

$\angle ABC = 90°$.

This result is illustrated for various positions of B in Figure 3.21.

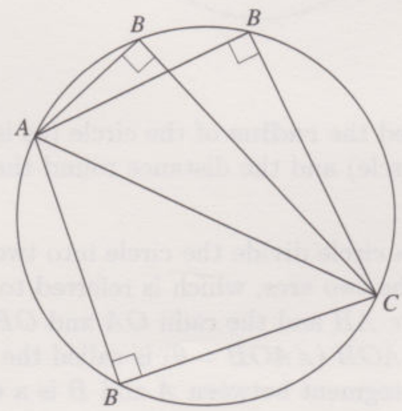


Figure 3.21

3.4 Areas

Area of a rectangle

> The area, A, of a rectangle is given by $A = l \times b = lb$, where l and b are its length and breadth respectively.

Area of a triangle

Corresponding to a chosen base, each triangle has a height, namely the perpendicular distance from the vertex opposite the base to the base (possibly extended). This is illustrated in Figure 3.22.

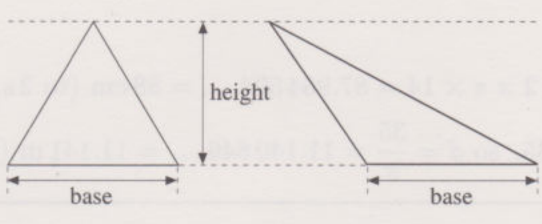


Figure 3.22

The area, A, of a triangle is given by

$$A = \frac{1}{2} \times b \times h = \frac{1}{2}bh,$$

where b is the length of a base and h is the corresponding height.

In the rectangle $AMND$ in Figure 3.19, reproduced below as Figure 3.23, $l = 3\,\text{cm}$, $b = 2\,\text{cm}$. So

$$\text{area of } AMND = 3 \times 2 = 6\,\text{cm}^2.$$

In the triangle AOB in Figure 3.23, $b = AB = 4\,\text{cm}$, $h = MO = \frac{3}{2}\,\text{cm}$. So

$$\text{area of } AOB = \frac{1}{2} \times 4 \times \frac{3}{2} = 3\,\text{cm}^2.$$

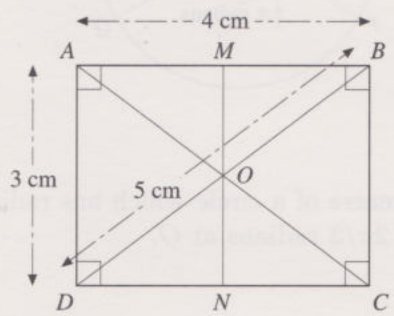


Figure 3.23

Exercise 3.6

Find the areas of the following shapes in Figure 3.23.

(a) rectangle $ABCD$

(b) triangle DBC

(c) triangle BOC

(d) triangle DON

Area of a circle

The area, A, of a circle is given by $A = \pi r^2$, where r is the radius.

Example 3.6

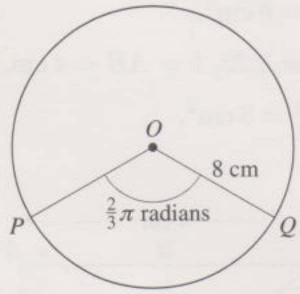


Figure 3.24

In Figure 3.24, O is the centre of a circle which has radius 8 cm. The arc PQ subtends an angle of $2\pi/3$ radians at O.

Find

(a) the area of the circle;

(b) the area of the sector POQ.

Solution

(a) The area of the circle is $\pi r^2 = \pi(8)^2 = 64\pi = 201.1\,\text{cm}^2$ (to 1 d.p.).

(b) $\angle POQ = 2\pi/3$, which is $\frac{1}{3}$ of a full turn. So the area of the sector POQ is $\frac{1}{3}\pi r^2 = \frac{1}{3}(64\pi) = 67.0\,\text{cm}^2$ (to 1 d.p.).

Exercise 3.7

The arc AB of a circle with centre O and radius 30 mm subtends an angle of 135° at O. Find, correct to 1 decimal place,

(a) the area of the circle; (b) the area of the sector AOB.

Module 4 Coordinates and lines

4.1 Points and Cartesian coordinates

Cartesian coordinates are used to specify the position of a point on a graph, relative to two axes which are at right angles to each other. In mathematical graphs, the horizontal axis is usually labelled the ***x*-axis**, and the vertical axis the ***y*-axis**.

Cartesian coordinates are often referred to simply as 'coordinates'.

'Axes' is the plural of 'axis'.

The coordinates of a point P are defined as follows.

Let PA and PB be the perpendiculars from P to the axes, as shown in the Figure 4.1.

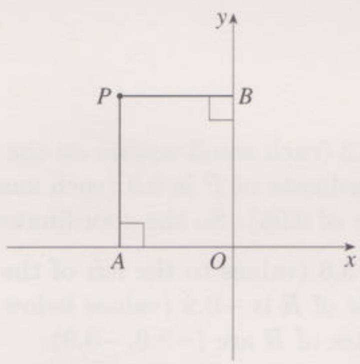


Figure 4.1

The signed distance from O, the **origin**, to A is denoted by a (a is positive if A is to the right of the y-axis, and negative if it is to the left). Similarly, the signed distance from O to B is denoted by b (b is positive if B is above the x-axis, and negative if it is below). The **ordered pair** of numbers (a, b) are the **coordinates** of P; a is the ***x*-coordinate**, b is the ***y*-coordinate**.

The origin, O, has coordinates $(0, 0)$.

Example 4.1

Write down the coordinates of the points P and R which are shown below.

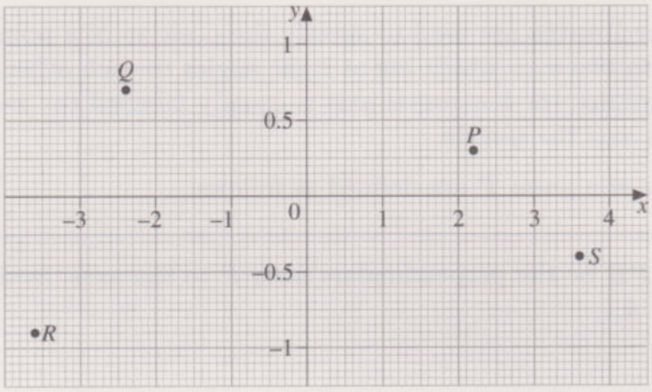


Figure 4.2

Solution

The x-coordinate of P is 2.2 (each small square on the x-axis represents a distance of 0.1). The y-coordinate of P is 0.3 (each small square on the y-axis represents a distance of 0.05). So the coordinates of P are $(2.2, 0.3)$.

The x-coordinate of R is -3.6 (values to the left of the y-axis are negative). The y-coordinate of R is -0.9 (values below the x-axis are negative). So the coordinates of R are $(-3.6, -0.9)$.

Exercise 4.1

(a) Write down the coordinates of the points Q and S in the figure in Example 4.1.

(b) On the same axes, plot the following points.

(i) $A\,(2.6, 1.1)$ (ii) $B\,(-3.1, 0.7)$ (iii) $C\,(-2.2, 1.125)$
(iv) $D\,(-2.2, -0.7)$

It is often convenient to write the coordinates of a point immediately after its label.

4.2 Lines, gradients and intercepts

Any two distinct points can be joined to give a straight line and, provided that the line is not parallel to the y-axis, its **gradient**, or **slope**, can be calculated from the coordinates of the two points. The value of x at which the line crosses the x-axis is called its **x-intercept**; the value of y at which the line crosses the y-axis is called its **y-intercept**.

Example 4.2

Find the gradient of the straight line CD in the figure below and give the x- and y-intercepts of the line.

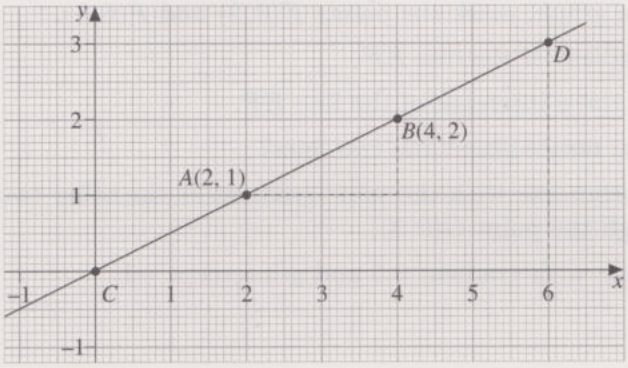


Figure 4.3

Solution

Any two distinct points on the line can be used to find the gradient.

For example, using the points $A(2,1)$ and $B(4,2)$, the **horizontal run** is $(4-2)$, which is 2 units, and the **vertical rise** is $(2-1)$, which is 1 unit.

The gradient is given by

$$\frac{\text{vertical rise}}{\text{horizontal run}} = \frac{1}{2} = 0.5.$$

Alternatively, using the two points $C(0,0)$ and $D(6,3)$, we have

$$\frac{\text{rise}}{\text{run}} = \frac{3-0}{6-0} = \frac{3}{6} = \frac{1}{2} = 0.5.$$

It is convenient to drop the words 'vertical' and 'horizontal' here.

The gradient of a straight line is a constant. Whichever two points are chosen on the line CD, the gradient evaluates to 0.5.

The line cuts the x-axis at the point $C(0,0)$, so the x-intercept is 0; it cuts the y-axis at the same point, so the y-intercept is 0.

Note that for every point on this line the y-coordinate is half or 0.5 times the x-coordinate. This relationship enables us to specify the line as an equation: the equation of this line is $y = 0.5x$ or $y = \frac{1}{2}x$.

Example 4.3

On graph paper draw and label axes from -2 to 6 on both the x- and y-axes.

It is common practice to write, for example, 'the line $y = 0.5x$' rather than 'the line whose equation is $y = 0.5x$'.

(a) Draw and label the line $y = 0.5x$.

(b) On the same graph, mark the points $E(0, 2)$ and $F(6, 5)$.

(c) Find the gradient of the line EF.

(d) Write down the y-intercept of the line EF.

(e) What do you notice about the two lines?

(f) Draw in another line with the same gradient as the other two lines but going through the point $G(0, 3)$.

(g) Draw in a line with gradient 0.5 and y-intercept -1.

Solution

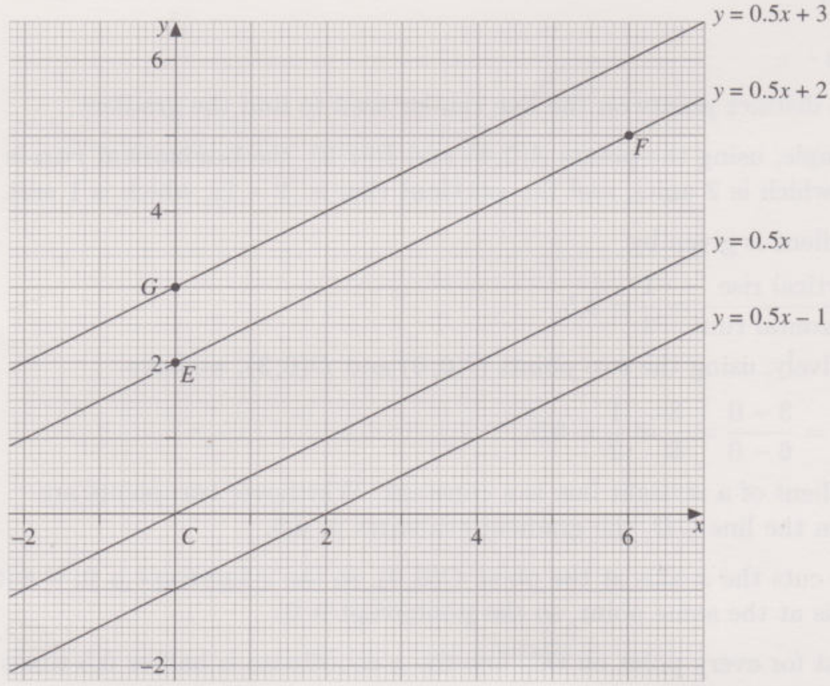


Figure 4.4

(a) On graph.

(b) On graph.

(c) Using the coordinates of $E(0, 2)$ and $F(6, 5)$, we have

$$\frac{\text{rise}}{\text{run}} = \frac{5-2}{6-0} = \frac{3}{6} = \frac{1}{2} = 0.5.$$

(d) The y-intercept is 2.

(e) They have the same gradient (and so are parallel to each other) but the line EF is shifted up two units.

(f) On graph.

(g) On graph.

Comment

The set of parallel lines in the figure above have been labelled with their equations. Each equation has the form

$$y = 0.5x + \text{the value of the } y\text{-intercept.}$$

The value of the y-intercept tells you by how much up or down the line $y = 0.5x$ has been shifted.

Exercise 4.2

Draw a graph with a scale from -2 to 4 on both the x- and y-axes.

Draw each of the following lines. Work out the gradient and y-intercept for each line.

(a) $y = x$ (This equation means that the x- and y-coordinates are the same for every point on the line.)

(b) $y = x + 1$

(c) $y = x - 1$

(d) $y = x + 0$

Lines with different gradients

The line $y = 0.5x$ has a gradient of 0.5 and the line $y = x$ has a gradient of 1. (It might help to remember that x is the same as $1x$.)

Exercise 4.3

Draw a graph with a scale from -4 to 4 on both the x- and y-axes.

Draw each of the following lines. Find the gradient of each line.

(a) $y = x$

(b) $y = 2x$ (This equation means that the y-coordinate is always double the x-coordinate.)

(c) $y = 0.25x$

(d) $y = -x$

From Exercise 4.3, it can be seen that the number multiplying x is the gradient, and the higher this number the steeper the line. If this number (often denoted by the letter m) is positive, the line slopes from bottom left to top right, and if it is negative, the line slopes in the opposite direction. If it is zero, the line is horizontal.

In general the equation of any straight line is of the form:

$$y = (\text{gradient multiplied by } x) \text{ added to the value of the } y\text{-intercept,}$$

that is

$$y = mx + c, \qquad \text{where } m \text{ is the gradient and } c \text{ is the } y\text{-intercept.}$$

Module 5 Algebra

5.1 Introduction

Algebra is concerned with generalised mathematical thinking arising from patterns and relationships. It is about expressing and interpreting these in words, symbols, diagrams and graphs. It is also concerned with:

- expressing the same thing in different forms;
- 'doing and undoing' (**inverse processes**).

The study and expression of generalities has a long history in problem-solving but it was not until the sixteenth century that symbols and letters were used as shorthand. In symbolic algebra, letters are used:

- to represent an 'as yet unknown' number, as in $5 = x + 3$;
- to represent variables, as in the relationship $y = x + 2$ (where the value of y is dependent on the value of x).

Algebra can be thought of as **generalised arithmetic**. An understanding of numerical processes, such as fractions, powers, factorising and simplifying, enables the same processes to be used with algebraic expressions using a mixture of numbers and letters. In order to understand algebra, it is important to be fluent in arithmetic (see Module 1).

5.2 Expressions

A collection of symbols such as $10 - 4N$, $x - 12$, $5a^2 + 2b + 4$ or $\frac{9}{5}C + 32$ is called an **algebraic expression**, or simply an expression.

Note that $-4N$ is an abbreviated way of writing $-4 \times N$.

Expressions like these are built up by adding a number of **terms** (which may be positive or negative). In the first expression above there are two terms: 10 and $-4N$.

An algebraic expression contains one or more symbols, such as N, x or c in the expressions above, standing for unspecified numbers: these symbols are often called **variables**.

The number by which a variable is multiplied is called its **coefficient**. In the expressions above the coefficient of N is -4, the coefficient of x is 1, that of a^2 is 5 and that of C is $\frac{9}{5}$.

Any term in an expression which is just a number is called a **constant**. The constants in the above expressions are 10, -12, 4 and 32.

Exercise 5.1

What are the variables, the coefficients and the constants in the following expressions?

(a) $4x + 3y + 5$ (b) $5k - 3n - 7$ (c) $\dfrac{a}{2} - \dfrac{2}{3}$

Forming algebraic expressions

When working on a mathematical problem, it is often necessary to translate words into algebraic expressions.

Example 5.1

The following table shows how a 'think of a number' problem may be translated into symbols. The symbol N stands for the number thought of.

Word expression	Algebraic expression	Simplified
Think of a number	N	N
double it	$2N$	$2N$
add 5	$2N+5$	$2N+5$
double the result	$2(2N+5)$	$4N+10$
subtract 2	$2(2N+5)-2$	$4N+8$
divide by 4	$\frac{1}{4}(2(2N+5)-2)$	$N+2$
take away the number you first thought of	$\frac{1}{4}(2(2N+5)-2)-N$	$(N+2)-N=2$

It can be seen from the final expression that whatever value of N is chosen the outcome will be 2.

Exercise 5.2

Complete the following tables, each of which represents a 'think of a number' problem. Use brackets where appropriate.

(a)

Word expression	Algebraic expression
Think of a number	p
triple it	
add 10	
double the result	
subtract 8	

(b)

Word expression	Algebraic expression
Think of a number	
square it	
multiply by 4	
add twice the number you first thought of	
add 8	

(c)

Word expression	Algebraic expression
Think of a number	x
	$x-2$
	$(x-2)^2$
	$(x-2)^2-4x$
	$(x-2)^2-4x+4$

Substitution in expressions

Values of an expression can be found by **substituting** numbers for symbols.

Example 5.2

(a) Find the value of $4N + 8$ when $N = 5, -3, 0$.

(b) Find the value of $2x^2 - 3$ when $x = -2$.

Remember that 'BIDMAS' (see Module 1) applies.

Solution

(a) When $N = 5$, $4N + 8$ becomes $4 \times 5 + 8 = 20 + 8 = 28$.

When $N = -3$, $4N + 8$ becomes $4 \times -3 + 8 = -12 + 8 = -4$.

When $N = 0$, $4N + 8$ becomes $4 \times 0 + 8 = 0 + 8 = 8$.

(b) When $x = -2$, $2x^2 - 3$ becomes

$$2 \times (-2)^2 - 3 = 2 \times 4 - 3 = 8 - 3 = 5.$$

Exercise 5.3

(a) Find the value of each of the following when $a = 7$.

(i) $4(a + 10)$ (ii) $4a + 10$ (iii) $a + 4 \times 10$

(b) Find the value of $\frac{2}{3}(4 + 3q)$ when $q = \frac{4}{5}$.

Simplifying expressions

Algebraic expressions are **simplified** by collecting together like terms.

Example 5.3

(a) Simplify the expression $a + a^2 - 5a + 4a$.

(b) Simplify the expression $x^2 - 4xy + 4y^2 + 6x^2 + 5xy + y^2$.

Solution

(a) This expression contains terms in a^2 and a. It simplifies to a^2, because the terms in a add to 0.

(b) This expression contains terms in (or multiples of) x^2, xy and y^2. These different types of term cannot be combined. But all the multiples of x^2 can be added together. The same applies to all the xy terms, and all the y^2 terms. Collecting these like terms together gives

$$x^2 + 6x^2 - 4xy + 5xy + 4y^2 + y^2.$$

The expression simplifies to $7x^2 + xy + 5y^2$.

Exercise 5.4

Simplify each of the following expressions as far as possible.

(a) $6B + 7B + 8B$ (b) $4x + 2 \times 3x$ (c) $11y - 5y$ (d) $6A + 7A^2$

(e) $x + 8x + 3x^2 - 2x^2$ (f) $x + 8x^2 + 3x - 2x^2$ (g) $-t + \frac{1}{2}t$

(h) $2a - b^2 + a + 2b^2 + a$

Expanding

Brackets are used in expressions to avoid ambiguity: for example, '$a - 1$ all squared' is written $(a-1)^2$.

Expressions may also be written as factors: for example, $3(a - 2b)$ indicates two terms with a common factor of 3. To **expand** this to find the unfactorised expression it is necessary to multiply each term in the bracket by the factor outside it. So $3(a - 2b) = 3 \times a + 3 \times (-2b) = 3a - 6b$.

Example 5.4

In each of the following expressions **multiply out** the brackets then simplify (collect like terms).

(a) $7(x + 3)$ (b) $b(3d + b)$ (c) $\frac{1}{2}(2k + 4) - 3k$

Solution

(a) $7(x + 3) = 7 \times x + 7 \times 3 = 7x + 21$

(b) $b(3d + b) = b \times 3d + b \times b = 3bd + b^2 = b^2 + 3bd$

(It is usual to put terms in power order and variables in alphabetical order.)

(c) $\frac{1}{2}(2k + 4) - 3k = \frac{1}{2} \times 2k + \frac{1}{2} \times 4 - 3k = k + 2 - 3k = -2k + 2$

Exercise 5.5

In each of the following expressions multiply out the brackets and simplify.

(a) $5(2 + x)$ (b) $x(a + b)$ (c) $\frac{3}{4}(a - 4b)$

(d) $a(a - 2) + a(a + 2)$ (e) $2x(y - x + 3) + 3xy$

(f) $a(b - a) + b(b - a)$

There are occasions when an expression arises as the product of two brackets; for example, $(a - b)(a + b)$. To expand (undo) these it is necessary to multiply every term in the second bracket by every term in the first. This is not difficult in itself but it is necessary to be systematic in order not to miss anything. The actual method does not matter, find one which suits you and practise until you are fluent.

Example 5.5

Expand each of the following expressions.

(a) $(a + b)(a + b)$ (b) $(2x + y)(x + 3y)$ (c) $(3r + 2s)(2r - 3s)$

Solution

(a) This solution illustrates one method of expanding brackets.

Take the 1st term of the 1st bracket	a	
Multiply by the 2nd bracket	$a(a+b) \;=\; a^2 + ab$	
Take the 2nd term of the 1st bracket	$+b$	
Multiply by the 2nd bracket	$+b(a+b) = \qquad ab + b^2$	ab is the same as ba.
Add	$a^2 + 2ab + b^2$	

Thus

$$(a+b)(a+b) = a^2 + 2ab + b^2.$$

(b) Another method is illustrated in Figure 5.1.

Think of the two expressions as being sides of a rectangle and work out the areas of each of the four sections.

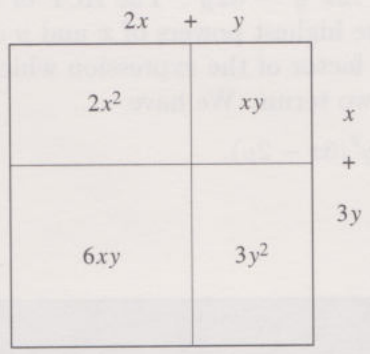


Figure 5.1

Then add the 'areas': $2x^2 + xy + 6xy + 3y^2 = 2x^2 + 7xy + 3y^2$.

Thus

$$(2x+y)(x+3y) = 2x^2 + 7xy + 3y^2.$$

(This geometric method can also be used with expressions involving negatives, but is not then quite so straightforward.)

(c) Figure 5.2 shows yet another method!

multiply 1st by 1st, add 2nd by 2nd	$(3r+2s)(2r-3s)$	$6r^2 - 6s^2$
multiply 1st by 2nd, add 2nd by 1st	$(3r+2s)(2r-3s)$	$-9rs + 4rs$
add together	$(3r+2s)(2r-3s)$ $(3r+2s)(2r-3s)$	$6r^2 - 5rs - 6r^2$

Figure 5.2

Exercise 5.6

In each of the following cases, expand the brackets and simplify the resulting expression.

(a) $(a+1)(a-1)$ (b) $(y+7)(y+1)$ (c) $(p+1)^2$

(d) $(2x+7)(x+2)$ (e) $(x-5)(x+3)$ (f) $(5x-2)(2x-3)$

(g) $(3a+4)(2a-3)$ (h) $(3a-4)(2a+3)$ (i) $(r-s)^2$

(j) $(a+b)^2$ (k) $(a+b)(a-b)$ (l) $(a-b)(a+b)$

Factorising

This is the inverse process of expanding – you need to learn to be able to spot likely factors. It is a very important skill to acquire and can seem quite difficult at first. Do persevere as it really is one which becomes easier with practice.

Consider the expression $12x^2y^2 - 8xy^3$. The HCF of 8 and 12 is 4. The quantity xy^2 contains the highest powers of x and y common to both terms. Thus $4xy^2$ is the factor of the expression which contains all the common factors of the two terms. We have

$$12x^2y^2 - 8xy^3 = 4xy^2(3x - 2y).$$

Example 5.6

Factorise each of the following expressions in the way described above.

(a) $3a - 12b$ (b) $5a^3b^2 + 5ab^3$

Solution

(a) 3 is the highest common factor of 3 and 12, and there is no other common factor. So

$$3a - 12b = 3(a - 4b).$$

(b) The highest common factor of 5 and 5 is 5. The quantity ab^2 contains the highest powers of a and b common to both terms. So we factor out $5ab^2$ to give

$$5a^3b^2 + 5ab^3 = 5ab^2(a^2 + b).$$

The difference of two squares

The result

$$a^2 - b^2 = (a+b)(a-b)$$

is an important one to learn. You need to be able to recognise the difference of two squares and know that it can be written as the product of the sum and difference of the two numbers.

Exercise 5.7

Factorise each of the following expressions as completely as possible. (Check your results by expanding them.)

(a) $4a^2 - 12b$ (b) $x^2 + x$ (c) $4p^2 + 4pq$ (d) $2\pi r - 2\pi$

(e) $12xy + 4xyz$ (f) $\frac{3\pi}{2}r - \frac{\pi}{2}$ (g) $8x^3 - x^2$ (h) $4x^4 + 2x^2$

(i) $p^2 - q^2$ (j) $4x^2 - y^2$ (k) $9s^2 - 4t^2$ (l) $\pi r^2 - 9\pi$

Not only can the difference of two squares be factorised into the product of two expressions but, as you will have noticed from the section on 'expanding', so can some other expressions (see, for example, Exercise 5.6, part (j): $a^2 + 2ab + b^2 = (a + b)(a + b)$).

Example 5.7

(a) Factorise each of the following expressions into the form $(x + a)(x + b)$ where x is a variable and a and b are constants (numbers) which can be positive, negative or zero.

(i) $x^2 + 5x + 6$ (ii) $x^2 - 10x + 16$

(b) Factorise $2x^2 + 7x - 4$ into the form $(2x + c)(x + d)$.

Solution

(a) Solutions can be found by 'experimentation' but a more systematic approach is to compare each given expression to the expansion of the general form $(x + a)(x + b)$. Now

$$(x + a)(x + b) = x^2 + ax + bx + ab = x^2 + (a + b)x + ab,$$

so we compare the general form $x^2 + (a + b)x + ab$ with each of the expressions.

(i) For $x^2 + 5x + 6$, $a + b = 5$ and $ab = 6$. So a and b are a pair of numbers which multiply together to give 6 and sum to 5. The pairs of factors of 6 are 1 and 6, 2 and 3, so $a = 2$ and $b = 3$ since 2 and 3 sum to 5. Therefore

$$x^2 + 5x + 6 = (x + 2)(x + 3).$$

(Check by expanding.)

(ii) For $x^2 - 10x + 16$, $a + b = -10$, $ab = 16$. Factors of 16 which sum to 10 are $2, 8$. But in this case the sum is negative, so we require $a = -2$ and $b = -8$. Therefore

$$x^2 - 10x + 16 = (x - 2)(x - 8).$$

(b) $2x^2 + 7x - 4$ is more difficult because the coefficient of x^2 is 2, so factors must be of the form

$$(2x + c)(x + d) = 2x^2 + 2xd + cx + cd = 2x^2 + (2d + c)x + cd.$$

So $2d + c = 7$, $cd = -4$. The pairs of factors of 4 are 1 and 4, 2 and 2, but for $cd = -4$ one of c and d must be negative and one positive. To satisfy $2d + c = 7$, we require $c = -1$, $d = 4$.

Therefore

$$2x^2 + 7x - 4 = (2x - 1)(x + 4).$$

Exercise 5.8

(a) Factorise each of the following expressions and check your result by expanding.

(i) $x^2 + 13x + 30$ (ii) $x^2 - 5x + 4$ (iii) $x^2 + 4x - 12$

(iv) $x^2 - 4x - 12$ (v) $4x^2 - 9$ (vi) $3x^2 + 11x - 4$

These expressions all involve x, but remember any letter might be used.

(b) Factorise those of the following expressions which can be factorised. Identify those that cannot.

(i) $x^2 + x - 2$ (ii) $x^2 - x + 2$ (iii) $x^2 - 6x$

(iv) $x^2 - 12$ (v) $2x^2 + 5x + 2$ (vi) $3x^2 + 3x - 6$

Rearranging expressions

When dealing with algebraic expressions it is often easier if they are rewritten in (rearranged in or manipulated into) different forms. This is particularly true of expressions involving fractions or roots. Being able to recognise such rearrangement possibilities can in some contexts make handling expressions much simpler.

Example 5.8

(a) Rearrange $\dfrac{p+q}{pq}$ as the sum of two fractions and simplify.

(b) Expand, simplify or otherwise manipulate the following into a single fraction.

$$\frac{3}{\sqrt{x}} + \sqrt{x}.$$

Solution

(a)
$$\frac{p+q}{pq} = \frac{p}{pq} + \frac{q}{pq} = \frac{1}{q} + \frac{1}{p}$$

(b)
$$\frac{3}{\sqrt{x}} + \sqrt{x} = \frac{3}{\sqrt{x}} + \sqrt{x}\frac{\sqrt{x}}{\sqrt{x}} = \frac{3 + \sqrt{x}\sqrt{x}}{\sqrt{x}} = \frac{3+x}{\sqrt{x}}$$

Exercise 5.9

(a) Rearrange each of the following expressions as the sum or difference of two fractions and simplify.

(i) $\dfrac{xy + yz}{xyz}$ (ii) $\dfrac{x^2 - 6}{2}$ (iii) $\dfrac{4ab - 2bc}{2b}$

(b) Simplify each of the following expressions.

(i) $(x - \sqrt{3})(x + \sqrt{3})$ (ii) $\dfrac{6x}{\sqrt{4x}}$ (iii) $\dfrac{\sqrt{3x}}{\sqrt{3y}}$

Algebraic fractions (rational functions)

Fractions involving expressions in x should be manipulated according to exactly the same rules as those for numerical fractions. For example, to combine

$$\frac{1}{x+1}-\frac{1}{x+3},$$

we use a common denominator – in this case $(x+1)(x+3)$ – and we begin by writing both fractions with this denominator.

Example 5.9

Simplify $\dfrac{1}{x+1}-\dfrac{1}{x+3}$.

Solution

We combine the terms in the following way.

$$\begin{aligned}\frac{1}{x+1}-\frac{1}{x+3}&=\frac{x+3}{(x+1)(x+3)}-\frac{x+1}{(x+1)(x+3)}\\&=\frac{(x+3)-(x+1)}{(x+1)(x+3)}\\&=\frac{x+3-x-1}{(x+1)(x+3)}\\&=\frac{2}{(x+1)(x+3)}\end{aligned}$$

Exercise 5.10

Simplify each of the following expressions.

(a) $\dfrac{1}{2}+\dfrac{1}{x}$ (b) $\dfrac{1}{x-4}+\dfrac{1}{3x-2}$ (c) $\dfrac{3x+2}{x^2+1}-\dfrac{2x+1}{x+2}$

5.3 Equations

An **equation** is any statement in which two algebraic expressions are related with an equals sign, or in which an expression equals a numerical value (including 0). For example, $3y+9=0$, $x^2+3=y-1$, or $x-y=7$. All the equations we are concerned with involve at most two variables or 'unknowns'. Finding values for the 'unknown(s)' in an equation is known as **solving** the equation.

Classifying equations

One way that equations are classified is by the highest power involved.

◇ **Linear equations** are ones where the highest power of any variable is 1; examples are $x + y = 3$, $x = 4$. Such equations can be represented graphically by **straight lines**. The general form is $y = ax + b$ (often written as $y = mx + c$).

◇ **Quadratic equations** are ones where the highest power of one, and only one, of the two variables is 2; an example is $y = x^2 - 4$. Such equations can be represented graphically by **parabolas**. The general form is $y = ax^2 + bx + c$ (where $a \neq 0$).

The parabola $y = x^2 - 4$ is shown in Figure 5.3.

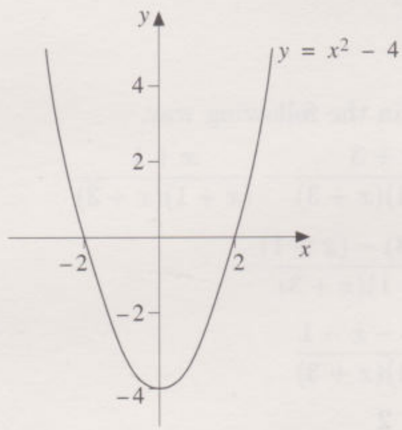


Figure 5.3

◇ Examples of higher order equations are

$$y = x^3 \text{ (cubic)}, \; y = x^4 - 1 \text{ (quartic)}, \; y = x^5 + x \text{ (quintic)}.$$

Linear equations

A linear equation can be solved provided there is only one 'unknown'.

Example 5.10

Solve the following linear equation $12 = 5 + \dfrac{x}{21}$.

Solution

We want 'x = a number'.
By swapping the two sides of the equation, the x term is put on the left-hand side:

$$5 + \frac{x}{21} = 12.$$

Subtracting 5 from each side (undoing the addition) gives

$$\frac{x}{21} = 12 - 5 = 7.$$

Multiplying each side by 21 (undoing the division) gives

$$x = 7 \times 21 = 147.$$

Exercise 5.11

Solve each of the following equations by undoing each operation in turn to obtain '$x =$ a number'.

(a) $2x + 1 = 5$ (b) $\frac{1}{2}x - 1 = \frac{1}{2}$ (c) $7x - 3 = 0$ (d) $3x = x + 4$

(e) $x + 2 = 2x - 1$ (f) $3(x + 1) = 2x - 1$

The graph of any equation of the form $y =$ constant is a straight line parallel to the x-axis; the graph of $x =$ constant is a straight line parallel to the y-axis. Examples are shown in Figure 5.4.

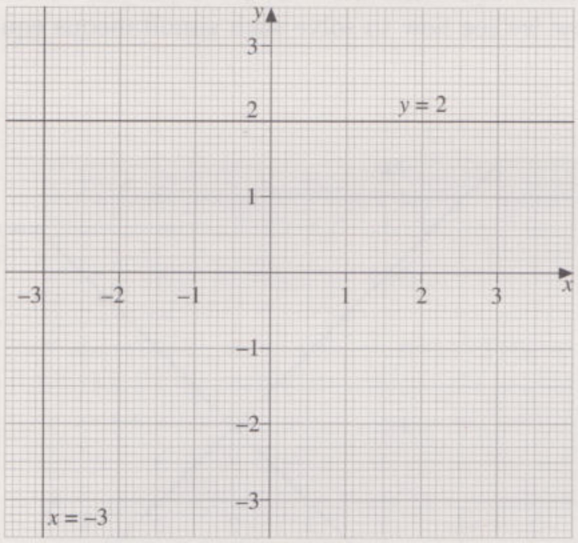


Figure 5.4

An equation in two unknowns, x and y, which can be arranged in the form $ax + by + c = 0$, where a, b and c are constants with $a \neq 0$ and $b \neq 0$, is called a linear equation in two unknowns. It can be plotted as a straight line graph.

This general form can be rearranged as

$$y = -\frac{a}{b}x + \left(-\frac{c}{a}\right).$$

Comparing this with $y = mx + c$ (see Module 4), we see that the line has gradient $-a/b$ and y-intercept $-c/a$.

Simultaneous equations

If there are two linear equations, in two unknowns, there is, in general, one pair of values for the unknowns which satisfies both equations. This pair of values corresponds to the coordinates of the point at which the two line graphs of the equations intersect. The equations are referred to as **simultaneous equations** because the solutions satisfy both equations.

The exceptions are when the lines are parallel or the same as each other.

For example, the lines corresponding to the simultaneous equations

$$y = x - 1$$
$$y = -x + 3$$

intersect at the point which has coordinates $(2, 1)$, as shown in Figure 5.5. So the solution of these equations is $x = 2$, $y = 1$ (as you can check by substitution). Below we see how to solve such simultaneous equations algebraically.

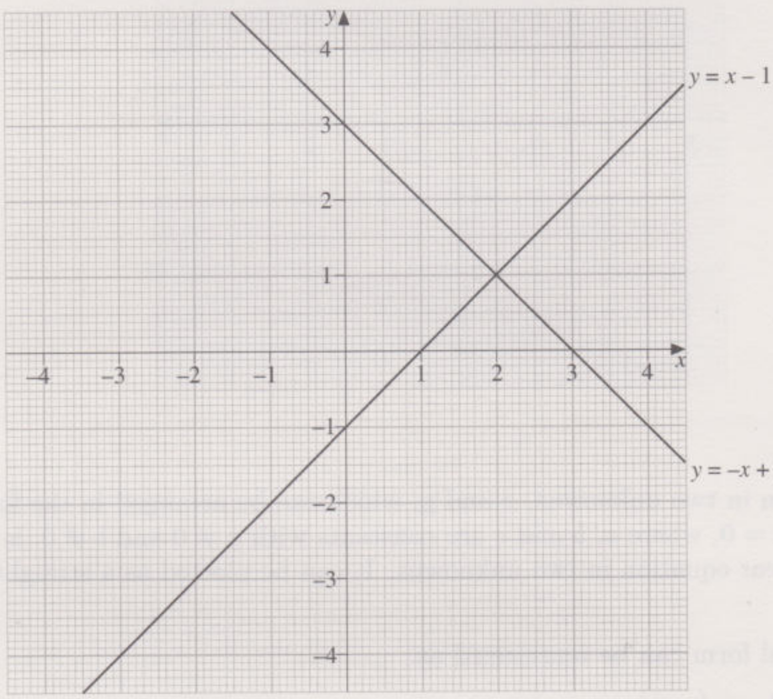


Figure 5.5

Pairs of simultaneous equations can be solved by arranging one equation so that one unknown is equal to an expression in terms of the other unknown, then substituting this expression in the second equation.

Note: There are other methods of solving simultaneous equations. For example, two equations may be solved using a method of elimination. See Solution 2 to Example 5.11 below.

If you are already confident with one particular method, you should continue to use it, but you might like to work through this method for comparison.

Example 5.11

Solve the following simultaneous equations.

The equations are labelled (5.1) and (5.2) for ease of reference.

$$3 + 5b = 2a \tag{5.1}$$
$$5 - b = a \tag{5.2}$$

Solution

(substitution method)

Look for easy substitutions:

$$a = 5 - b \quad \text{(rewriting Equation 5.2 so that } a \text{ is in terms of } b\text{).} \tag{5.3}$$

Then

$$3 + 5b = 2(5 - b) \quad \text{(substituting for } a \text{ in Equation 5.1)} \tag{5.4}$$

from which it follows that

$$3 + 5b = 10 - 2b \quad \text{(expanding the brackets in Equation 5.4)}$$

i.e. $3 + 7b = 10$ (rearranging).

It follows that

$$7b = 10 - 3 = 7;$$

hence $b = 1$, and since $a = 5 - b$ (Equation 5.3), $a = 5 - 1 = 4$.

The solution is $a = 4$ and $b = 1$. (It is good practice to check your solutions by substituting the values back in the original equations.)

(elimination method)

Look for a way to make the multiple of either variable the same in both equations. In this case, multiply Equation 5.2 by 5 so that b is eliminated.

$$3 + 5b = 2a \tag{5.5}$$
$$25 - 5b = 5a \tag{5.6}$$

Add the two quantities on each side of Equations 5.5 and 5.6 to give

$$28 = 7a$$

or $a = 4$.

Now we can substitute $a = 4$ in Equation 5.2 to find $b = 1$ (and check that $a = 4$, $b = 1$ satisfies Equation 5.1).

Alternatively, we could eliminate a. Multiply Equation 5.2 by -2 to give

$$3 + 5b = 2a \tag{5.7}$$
$$-10 + 2b = -2a. \tag{5.8}$$

Now we add the two quantities on each side of Equations 5.7 and 5.8:

$$-7 + 7b = 0$$

or $b = 1$.

The solution is $a = 4$, $b = 1$. (Whichever method you use to solve simultaneous equations, it is worth the effort of putting your solution back into each equation at the end, in order to check the result.)

Exercise 5.12

Solve each of the following pairs of simultaneous equations.

(a) $X = 7 + Y$
$2X = 2 + Y$

(b) $2x = y - 1$
$3x = y + 1$

(c) $y = x - 1$
$y = -x + 3$

(d) $x = 2y - 3$
$3x + 2y = 15$

(e) $3p - 3 = 3q$
$7p = 1 + 4q$

Quadratic equations

In general there are two values of x which satisfy the general quadratic equation $ax^2 + bx + c = 0$, represented by where the graph of the parabola $y = ax^2 + bx + c$ crosses the x-axis (i.e. when $y = 0$). For example, it is easy to see that the equation $x^2 - 4 = 0$ has solutions $x = -2$ and $x = 2$. These values are the x-intercepts of the graph of the parabola $y = x^2 - 4$, shown in Figure 5.6.

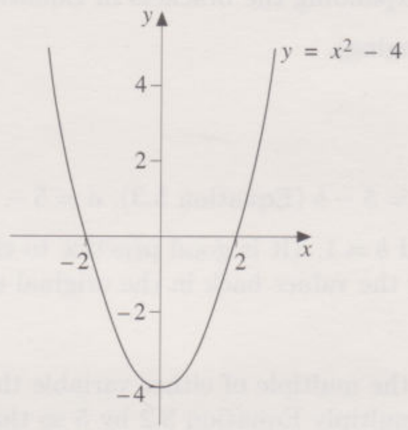


Figure 5.6

One way to solve a quadratic equation with real roots is to rearrange it in the form $ax^2 + bx + c = 0$ and then factorise the expression on the left-hand side. This will produce the product of two brackets, say $(px - q)(rx - s)$. If the value of either factor is 0, then the product will be 0. So putting each factor equal to 0 in turn, will produce two values of x.

$0 \times$ any number $= 0$.

> If a quadratic equation can be written in the form $(px - q)(rx - s) = 0$ then either $px - q = 0$ or $rx - s = 0$
>
> If $px - q = 0$, then $px = q$; so $x = q/p$.
>
> If $rx - s = 0$, then $rx = s$; so $x = s/r$. Hence the solutions to the quadratic equation
>
> $$(px - q)(rx - s) = 0$$
>
> are $x = \dfrac{q}{p}$ and $x = \dfrac{s}{r}$.

Example 5.12

Use factorisation to solve each of the following quadratic equations.

(a) $x^2 - 5x + 6 = 0$

(b) $x^2 + 6x + 9 = 0$

(c) $2y^2 - 9y - 5 = 0$

Solution

(a) $x^2 - 5x + 6 = 0$ factorises to give $(x-2)(x-3) = 0$ and so has the solutions $x = 2$ and $x = 3$.

(b) $x^2 + 6x + 9 = 0$ factorises to give $(x+3)(x+3) = 0$ and, because the two brackets are identical, $x = -3$ is the only solution.

(c) $2y^2 - 9y - 5 = 0$ factorises to give $(2y+1)(y-5) = 0$ and so has the solutions $y = -\frac{1}{2}$ and $y = 5$.

Exercise 5.13

Solve each of the following equations by factorisation.

(a) $x^2 + 5x + 6 = 0$ (b) $y^2 - 10y + 16 = 0$ (c) $2x^2 + 7x - 4 = 0$

(d) $4p^2 + 12p + 9 = 0$

Factorising is often the simplest way to solve a quadratic equation, but when this proves difficult there is a formula (**the quadratic formula**) to use. For the general equation $ax^2 + bx + c = 0$, the formula for the two solutions is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note: It is the $\pm$ which produces the two solutions and $\sqrt{}$ indicates the positive square root. If $b^2 - 4ac$ is negative, there are no real solutions because no negative number has a real square root.

Example 5.13

Use the quadratic formula to solve $6x^2 + 7x - 5 = 0$.

Solution

Comparing $6x^2 + 7x - 5 = 0$ to the general form $ax^2 + bx + c = 0$, gives $a = 6$, $b = 7$, $c = -5$.

Then substitute these values into the quadratic formula:

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4 \times 6 \times (-5)}}{2 \times 6} \\ &= \frac{-7 \pm \sqrt{49 + 120}}{12} \\ &= \frac{-7 \pm \sqrt{169}}{12} \\ &= \frac{-7 \pm 13}{12} \end{aligned}$$

So, taking the + sign, we have

$$x = \frac{-7 + 13}{12} = \frac{6}{12} = \frac{1}{2};$$

and taking the − sign, we have

$$x = \frac{-7 - 13}{12} = \frac{-20}{12} = -\frac{5}{3}.$$

Thus the two solutions are $x = \frac{1}{2}$ and $x = -\frac{5}{3}$.

(So the original equation $6x^2 + 7x - 5 = 0$ could have been factorised to $6(x - \frac{1}{2})(x + \frac{5}{3}) = 0$, or written another way, to $(2x - 1)(3x + 5) = 0$.)

Exercise 5.14

Use the quadratic formula to solve each of the following equations.

(a) $2x^2 + 7x - 8 = 0$ (b) $4p^2 + 12p + 9 = 0$

(c) $y^2 + y + 1 = 0$

Sometimes an equation for a variable may involve algebraic fractions. In such a case, the first step is to simplify the algebraic fractions before solving the equation that results.

Example 5.14

Solve the equation $\dfrac{1}{x+1} - \dfrac{1}{2x+1} = \dfrac{1}{6}$.

Solution

First we bring all the terms onto the left-hand side, rewriting the equation as $\dfrac{1}{x+1} - \dfrac{1}{2x+1} - \dfrac{1}{6} = 0$.

This time the common denominator is $6(x+1)(2x+1)$.

$$\begin{aligned}
\frac{1}{x+1} - \frac{1}{2x+1} - \frac{1}{6} &= \frac{6(2x+1)}{6(x+1)(2x+1)} - \frac{6(x+1)}{6(x+1)(2x+1)} - \frac{(2x+1)(x+1)}{6(x+1)(2x+1)} \\
&= \frac{6(2x+1) - 6(x+1) - (2x+1)(x+1)}{6(x+1)(2x+1)} \\
&= \frac{12x + 6 - 6x - 6 - 2x^2 - 3x - 1}{6(x+1)(2x+1)} \\
&= \frac{-2x^2 + 3x - 1}{6(x+1)(2x+1)} \\
&= -\frac{2x^2 - 3x + 1}{6(x+1)(2x+1)} \\
&= -\frac{(2x-1)(x-1)}{6(x+1)(2x+1)}
\end{aligned}$$

Hence $\dfrac{1}{x+1} - \dfrac{1}{2x+1} = \dfrac{1}{6}$ is equivalent to $-\dfrac{(2x-1)(x-1)}{6(x+1)(2x+1)} = 0$.

That is, $(2x-1)(x-1) = 0$, so $x = 1$ or $x = \frac{1}{2}$.

Exercise 5.15

Solve the equation $\dfrac{1}{x-4} + \dfrac{1}{3x-2} = -\dfrac{3}{4}$.

Exponential equations

Logarithms are used in solving equations in which the unknown is an exponent (power).

Example 5.15

Solve each of the following exponential equations.

(a) $5^x = 12$

(b) $25^{2x+1} = 15$

Solution

(a) Since the quantities 5^x and 12 are equal, so are their logarithms. Taking the common logarithm of each side of the equation gives

Here log is used rather than $\log_{10}$ for common logarithms.

$$\log 5^x = \log 12.$$

Then, using Rule 3 of logarithms that $\log a^n = n \log a$ (see Module 1), we obtain

$$x \log 5 = \log 12$$

so that

$$x = \frac{\log 12}{\log 5} = 1.54 \text{ (to 2 d.p.)}.$$

Alternatively, using natural logarithms, we have

$$\ln 5^x = \ln 12$$

so that

$$x \ln 5 = \ln 12,$$

from which

$$x = \frac{\ln 12}{\ln 5} = 1.54 \text{ (to 2 d.p.)}.$$

(b) Taking the natural logarithm of each side of $25^{2x+1} = 15$ gives

$$\ln 25^{2x+1} = \ln 15$$

so that, by Rule 3 of logarithms,

$$(2x+1) \ln 25 = \ln 15,$$

from which

$$2x + 1 = \frac{\ln 15}{\ln 25}.$$

Hence

$$2x = \frac{\ln 15}{\ln 25} - 1$$

so that

$$x = \frac{1}{2}\left(\frac{\ln 15}{\ln 25} - 1\right) = -0.079 \text{ (to 3 d.p.)}.$$

Exercise 5.16

Solve each of the following exponential equations, giving your answer correct to three decimal places.

(a) $3^x = 5$ (b) $5^x = 3$ (c) $12^{-x} = 5$ (d) $4^{x-1} = 4$

5.4 Inequalities

Inequalities and intervals

Inequalities are more easily understood and manipulated when you are fluent with the symbols, and reference is made to the number line (either actually or mentally) discussed in Module 1.

The following example explains how inequalities may be represented using intervals of the number line.

Example 5.16

(a) The inequality $\frac{1}{2} < x$ is equivalent to $x > \frac{1}{2}$, which is read as

'x is greater than a half'.

This means that x can take any value strictly greater than $\frac{1}{2}$, that is, any point on the number line to the right of $\frac{1}{2}$. This range of values of x is shown as an **interval** on the number line in Figure 5.7. The open dot shows that $\frac{1}{2}$ is excluded.

An **interval** is a range of values with no gaps.

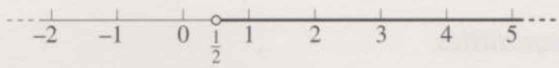


Figure 5.7

(b) The inequality $-4 < x \leq 23$ can be read as

'x is greater than minus 4 and less than or equal to 23'.

This means that x can take any of the values between -4 and 23, including 23 but excluding -4. The set of values of x which satisfies both inequalities can be represented as an interval on the number line shown in Figure 5.8, where the solid dot shows that 23 is included, and the open one that -4 is excluded.

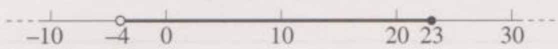


Figure 5.8

(c)

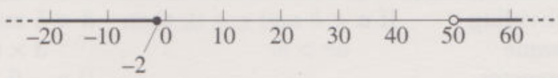


Figure 5.9

The inequality illustrated in Figure 5.9 is

'x is either less than or equal to -2, or greater than 50'

and is expressed symbolically as $x \leq -2$ or $x > 50$.

In case (c) above the description refers to two separate parts of the number line, and so two separate inequalities are needed. The first states that x can take the value -2 or any value to the left of -2, the second that x can take any value to the right of 50.

The fact that there is no point on the line which marks a boundary below -2 or above 50 could have been indicated by using the symbol ∞, for infinity, as follows:

$$-\infty < x \leq -2 \quad \text{or} \quad 50 < x < \infty.$$

Note: $\leq$ and $\geq$ are never used with ∞.

Exercise 5.17

(a) Write out each of the following inequalities in words.

(i) $180 < x < 360$ (ii) $a \geq 3$

(b) Express each of the following descriptions as inequalities.

(i) x is positive.

(ii) x is greater than or equal to minus 32.

(iii) a is a number between 1 and 10, but not including 10.

You may recall that this is the range of values a can be when a number is written in scientific notation, i.e. $a \times 10^n$ (see Module 1).

(c) Represent the inequalities, given in part (a) above, as intervals on the number line.

Rearranging inequalities

Inequalities can be rearranged and manipulated in similar ways to equations provided care is taken with the direction of the inequality signs. It helps to consider an algebraic and numerical example of each rule.

Rule	Example: Algebraic	Example: Numerical
1 Interchanging the sides of an inequality, reverses the direction of the inequality	If $a < b$, then $b > a$	$3 < 5$, so $5 > 3$
2 Adding the same number to, or subtracting the same number from both sides preserves the direction of the inequality	If $a > b$, then $a - c > b - c$ and $a + c > b + c$	$5 > 3$, so $5 - 2 > 3 - 2$ (i.e. $3 > 1$) and $5 + 2 > 3 + 2$ (i.e. $7 > 5$)
3 Multiplying or dividing both sides by the same *positive* number preserves the direction of the inequality	If $a > b$ and $c > 0$, then $ac > bc$ and $a/c > b/c$	$5 > 3$ and $2 > 0$, so $5 \times 2 > 3 \times 2$ (i.e. $10 > 6$) and $5/2 > 3/2$
4 Multiplying or dividing both sides by the same *negative* number reverses the direction of the inequality	If $a < b$ and $c < 0$, then $ac > bc$ and $a/c > b/c$	$3 < 5$ and $-2 < 0$, so $3 \times (-2) > 5 \times (-2)$ (i.e. $-6 > -10$) and $3/-2 > 5/-2$ (i.e. $-3/2 > -5/2$)

Using these rules it is possible to rearrange and simplify algebraic inequalities.

Example 5.17

Rewrite each of the following inequalities in the form: x 'inequality symbol' number, e.g. $x \geq 4$.

This process is called 'solving the inequality'.

(a) $3 < x + 1$ (b) $x - 5 > -7$ (c) $15 > 3x$

(d) $-4x < -12$ (e) $2(7 - x) \geq 22$

Solution

(a) $3 < x + 1$

$x + 1 > 3$ (to get x on the L.H.S.) (Rule 1)

L.H.S. denotes left-hand side.

$x > 3 - 1$ (to get x on its own) (Rule 2)

$x > 2$

(b) $x - 5 > -7$

$x > -7 + 5$ (adding 5) (Rule 2)

$x > -2$

(c) $15 > 3x$

$3x < 15$ (to get x on the L.H.S.) (Rule 1)

$x < 5$ (dividing by +3) (Rule 3)

(d) $-4x < -12$

$x > 3$ (dividing by −4) (Rule 4)

(e) $2(7 - x) \geq 22$

$14 - 2x \geq 22$ (expanding)

$-2x \geq 22 - 14$ (subtracting 14) (Rule 2)

$-2x \geq 8$

$x \leq -4$ (dividing by −2) (Rule 4)

Exercise 5.18

Use the rules to rewrite the following inequalities so that the left-hand side is x in each case.

(a) $3x + 7 \geq 39 - 5x$

(b) $\frac{1}{4} \geq \frac{1}{2}x$

(c) $7 < 3(5 + x)$

Module 6 Trigonometry

6.1 Trigonometric ratios

In Figure 6.1, the angle θ is common to the triangles ABC and DEC, both of which are right-angled. These triangles are scaled copies of each other; they are examples of **similar triangles**. The ratios of lengths

$$\frac{DE}{AB}, \frac{DC}{AC} \text{ and } \frac{EC}{BC}$$

are equal, their common value giving the scaling from ΔABC to ΔDEC.

Two similar triangles have the same three angles. Similar figures are considered in Module 8.

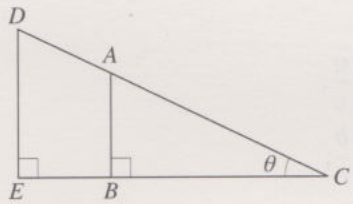


Figure 6.1

If two right-angled triangles are similar – like those in Figure 6.1 – then the ratio of the lengths of any one pair of sides in one triangle is equal to the ratio of the lengths of the corresponding sides in the other triangle. For example,

$$\frac{AB}{BC} = \frac{DE}{EC} \text{ and } \frac{BC}{CA} = \frac{EC}{CD}.$$

Such ratios are called the **trigonometric ratios**and can be used for calculating lengths and angles in right-angled triangles. The three basic trigonometric ratios are defined below.

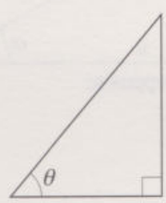


Figure 6.2

If θ is an acute angle in any right-angled triangle (see Figure 6.2), then the **sine** of θ, abbreviated to $\sin\theta$ is defined as

$$\sin\theta = \frac{\text{length of side opposite to } \theta}{\text{length of hypotenuse}};$$

the **cosine** of θ, abbreviated to $\cos\theta$, is defined as

$$\cos\theta = \frac{\text{length of side adjacent to } \theta}{\text{length of hypotenuse}};$$

the **tangent** of θ, abbreviated to $\tan\theta$, is defined as

$$\tan\theta = \frac{\text{length of side opposite to } \theta}{\text{length of side adjacent to } \theta}.$$

For convenience, we usually just write $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, for example.

These definitions are exemplified below for the triangle in Figure 6.3.

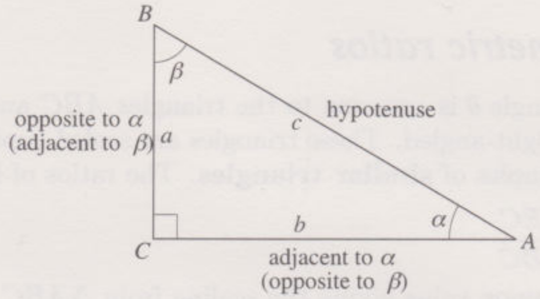


Figure 6.3

We have

Instead of using α, the sine of the angle at A may be written $\sin A$.

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c};$$
$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c};$$
$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}.$$

The hypotenuse is always the largest side. The opposite and adjacent sides depend on the angle chosen. For example, in Figure 6.4,

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}.$$

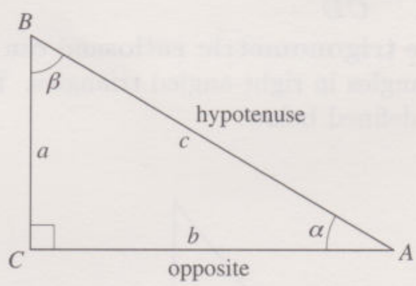


Figure 6.4

Angles are given sometimes in degrees and sometimes in radians. Your calculator will be able to handle both, but not at the same time. So it is important to check which mode (degrees or radians) your calculator is in at any particular time. In the next exercise you will use your calculator to find values of sines, cosines and tangents of various angles.

Exercise 6.1

Find out how to switch your calculator between **degrees** and **radians** (see also the section on angle measures in Module 2).

Defining sine, cosine and tangent for angles other than acute angles will be considered in Module 7.

(a) Make sure that your calculator is in degree mode. Use it to investigate the sines, cosines and tangents of angles such as

$$0°, 30°, 45°, 60°, 90°, 120°, 150°, 163°, 180°, 199°.$$

Draw up a table to display your results.

Then try the same angles, but with a negative sign in front. Again use a table to display your results.

(b) Now switch your calculator to radian mode. Also find the key which is used to input π. Investigate the sines, cosines and tangents of angles in radians, such as

$$0, 0.25, \frac{\pi}{6}, \frac{\pi}{4}, \frac{4\pi}{3}, 1.5, \frac{\pi}{2}, 2, \pi, 2\pi, 3\pi$$

and put the results in a table.

Try the same investigations with the negatives of the same angles, and display those results as well.

Comment

You should have found that your calculator gives an answer for any value you have input with [sin] or [cos]. Calculators respond to angles of any size, even those greater than a complete turn (i.e. angles greater than 360° or 2π radians) or those in a negative direction (i.e. angles less than 0).

However, some values used with [tan], such as 90° or $\pi/2$ radians, will have caused your calculator to give an error message. This is because the tangents of these angles are undefined.

You may have noticed that some trigonometric ratios are the same for different angles; for example, $\sin 30° = \sin 150° = 0.5$.

You will probably find that the second functions for the [sin], [cos] and [tan] keys are labelled [$\sin^{-1}$], [$\cos^{-1}$] and [$\tan^{-1}$].

Exercise 6.2

Set your calculator in degree mode and find the value of $\sin 30°$. Now use the second function for this key, [$\sin^{-1}$], and see what happens. Can you describe the effect of using [$\sin^{-1}$]? Try some more values. Does the same thing happen? Try the other two trigonometric function keys [cos] and [tan] with [$\cos^{-1}$] and [$\tan^{-1}$]. Does the same thing happen with them?

Now switch your calculator into radian mode and try the same thing again. Do you observe the same kind of effect?

Comment

$\sin^{-1} 0.5$ is sometimes denoted as $\arcsin 0.5$. Similarly, arccos is sometimes used for $\cos^{-1}$, and arctan for $\tan^{-1}$.

The calculator gives $\sin 30 = 0.5$ and using the second function returns $\sin^{-1} 0.5 = 30$: $\sin^{-1}$ 'undoes' sin, and sin 'undoes' $\sin^{-1}$.

'$\sin^{-1} 0.5 = 30$' is interpreted as 'the angle whose sine is 0.5 is equal to 30°'.

You may have noticed that, for example, $\cos 300° = 0.5$ but using the second function returns $\cos^{-1} 0.5 = 60$. This is because the $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ keys return angles in certain ranges. (The ranges usually are: $-90°\ (-\pi/2)$ to $90°(\pi/2)$, for $\sin^{-1}$; $0°\ (0)$ to $180°\ (\pi)$, for $\cos^{-1}$; $-90°\ (-\pi/2)$ to $90°\ (\pi/2)$ for $\tan^{-1}$.) So if you use [$\sin^{-1}$], [$\cos^{-1}$] or [$\tan^{-1}$] in the middle of solving a problem, you may have to refer back to the problem to decide whether or not the answer given by the calculator is sensible, and adjust it if necessary.

In the right-angled triangles we deal with here, such adjustments are never necessary.

When using radian mode, the main difference that you are likely to notice is that, although you may have entered an angle as an expression involving π, the answer will always be returned as a decimal number. (You can check if a decimal number is a multiple of π by dividing by π.)

Example 6.1

For triangles, angles may be identified by the vertex letters, as here.

(a) In the triangle ABC shown in Figure 6.5, $\angle C$ is a right angle, $\angle A = 35°$ and the length of the hypotenuse AB is 15 cm. Find the unknown angle and lengths. (This is often referred to as 'solving the triangle'.)

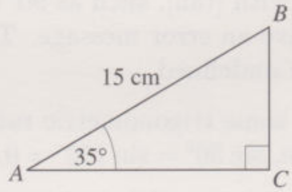


Figure 6.5

(b) In the triangle XYZ, $\angle Y$ is a right angle, and the lengths XY and YZ are 4 cm and 6 cm respectively. Find the unknown angles.

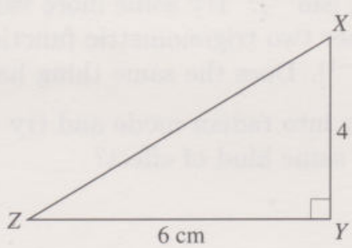


Figure 6.6

Solution

(a) (i) $\angle B = 180° - 90° - 35° = 55°$

(ii) $\dfrac{AC}{AB} = \cos 35°$, so

$AC = AB \cos 35° = 15 \cos 35° = 12.29\,\text{cm}$ (to 2 d.p.).

(iii) $\dfrac{BC}{AB} = \sin 35°$, so

$BC = AB \sin 35° = 15 \sin 35° = 8.60\,\text{cm}$ (to 2 d.p.).

(b) We have $\tan X = ZY/YX = 6/4 = 1.5$. So $\angle X$ is the angle whose tangent is 1.5, that is, $\angle X = \tan^{-1} 1.5 = 56.31°$ (to 2 d.p.). Then

$\angle Z = 90 - \angle X = 90° - 56.31° = 33.69°$ (to 2 d.p.).

Exercise 6.3

(a) For each of the following right-angled triangles, find all the unknown sides and angles.

(i)

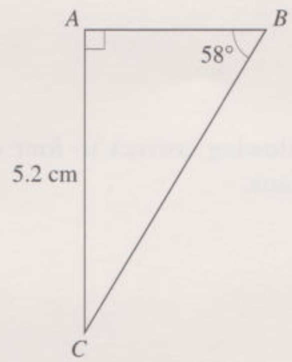


Figure 6.7

(ii)

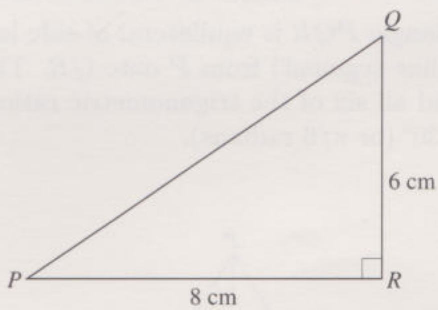


Figure 6.8

(iii)

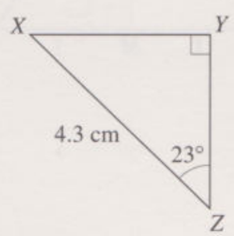


Figure 6.9

(b) A right-angled triangle has an angle of 63° and a hypotenuse of length 30 cm. Find the third angle and the lengths of the two shorter sides.

(c) The two shorter sides of a right-angled triangle are 5 cm and 12 cm. Find the unknown angles in radians correct to two decimal places.

Let θ be an acute angle. The reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are named and defined as follows.

cosecant of θ: $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

secant of θ: $\sec\theta = \dfrac{1}{\cos\theta}$

cotangent of θ: $\cot\theta = \dfrac{1}{\tan\theta}$

Note: These reciprocals of $\tan\theta$, $\cos\theta$ and $\sin\theta$ should not be confused with $\tan^{-1}$, $\cos^{-1}$, $\sin^{-1}$ which are the inverses of tan, cos and sin (i.e. the 'undoing' of each of the operations).

Exercise 6.4

(a) Evaluate each of the following correct to four decimal places.

(i) $\cot 40°$

(ii) $\sec 20°$

(iii) $\operatorname{cosec} 82°$

(b) Evaluate each of the following correct to four decimal places. The angles are given in radians.

(i) $\operatorname{cosec} 1.2$

(ii) $\cot \pi/5$

(iii) $\sec \pi/8$

Example 6.2

In Figure 6.10, the triangle PQR is equilateral of side length 2 units. PM is the perpendicular (line segment) from P onto QR. This perpendicular bisects QR at M. Find all six of the trigonometric ratios for angles of 60° (or $\pi/3$ radians) and 30° (or $\pi/6$ radians).

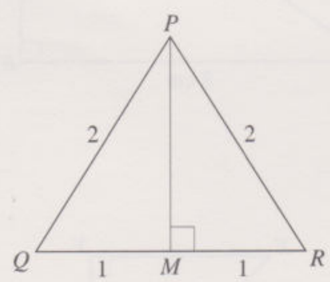


Figure 6.10

Solution

Angles $\angle PQM$ and $\angle PRM$ are both $60° = \pi/3$ radians, since triangle PQR is equilateral.

$\angle QPM = \angle MPR = 30° = \pi/6$ radians, since triangles PQM and PRM are right-angled. Also $QM = MR = 1$.

By Pythagoras' Theorem

$$PM^2 = PQ^2 - QM^2$$

so that

$$PM = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}.$$

The six trigonometric ratios for the angles $\pi/3$ and $\pi/6$ radians are therefore the following.

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2} = 0.5$$

$$\cos\frac{\pi}{3} = \frac{1}{2} = 0.5 \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}} \qquad \operatorname{cosec}\frac{\pi}{6} = \frac{2}{1} = 2$$

$$\sec\frac{\pi}{3} = \frac{2}{1} = 2 \qquad \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

$$\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}} \qquad \cot\frac{\pi}{6} = \sqrt{3}$$

In a complicated figure it is often necessary to use three letters to specify an angle. For example, in Figure 6.10 $\angle P$ is ambiguous, so we specify which angle is being referred to by writing $\angle QPM$ or $\angle MPR$.

The exact values for sin, cos and tan of $\frac{\pi}{3}$ and $\frac{\pi}{6}$ are very useful and worth remembering. As they will appear on your calculator as decimals, it is also worth noting that

$$\frac{\sqrt{3}}{2} = 0.866\,025\ldots,$$

and that

$$\sqrt{3} = 1.732\,05\ldots.$$

Exercise 6.5

The triangle XYZ in Figure 6.11 is both right-angled and isosceles, with $XY = ZY = 1$ unit.

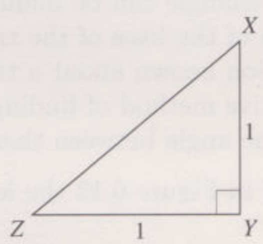


Figure 6.11

(a) Write down the length of XZ, and the sizes of $\angle Z$ and $\angle X$ in radians.

(b) Use this information to write down the exact values of the six trigonometric ratios for an angle of $\pi/4$ radians.

Exercise 6.6

(a) Evaluate the following pairs of ratios, to seven decimal places.

(i) $\sin 40°$ and $\cos 50°$ (ii) $\cos 12°$ and $\sin 78°$

(b) Using a suitable diagram, show that for positive angles A and B which are less than $90°$, if $A + B = 90°$ then $\sin A = \cos B$ and $\cos A = \sin B$.

Exercise 6.7

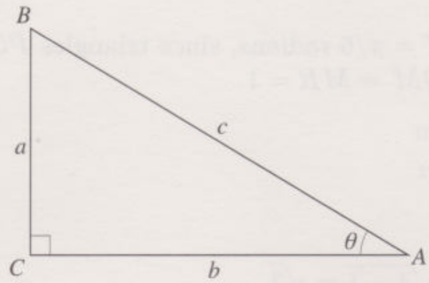


Figure 6.12

The squares of trigonometric ratios $(\sin\theta)^2$, $(\cos\theta)^2$, $(\tan\theta)^2$ are written as follows: $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$.

This is an important **trigonometric identity**.

Use Figure 6.12 to show that for $0 < \theta < \pi/2$,

$$\sin^2\theta + \cos^2\theta = 1,$$

by completing the following statements.

$\sin\theta =$

$\sin^2\theta =$

$\cos\theta =$

$\cos^2\theta =$

$$\sin^2\theta + \cos^2\theta =$$

$$= \frac{\quad + \quad}{\qquad}$$

Now think 'Pythagoras'!

$\sin^2\theta + \cos^2\theta =$

6.2 Area of a general triangle

It is known that the area of a triangle can be found using the formula $A = \frac{1}{2}bh$, where b is the length of the base of the triangle and h is its height. However, the information known about a triangle does not always include its height. An alternative method of finding the area of a triangle can be used if two sides and the angle between them are known.

The angle between two sides is known as the 'included angle'.

Suppose that in triangle XYZ in Figure 6.13 the lengths x and y and the angle Z are known.

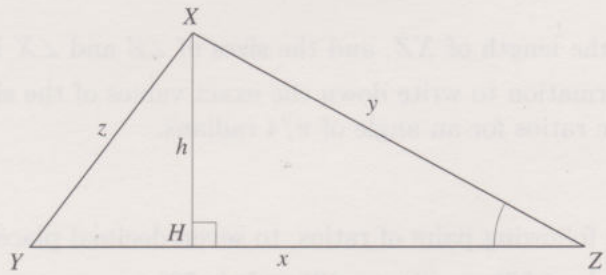


Figure 6.13

Taking YZ as the base, draw a line from X meeting YZ in a right angle at H. The length of this line, h, is the height of the triangle. From triangle XZH, we have $\sin Z = h/y$ so that $h = y \sin Z$.

Substituting this in $A = \frac{1}{2}bh$ gives $A = \frac{1}{2}xy \sin Z$.

> The area of triangle XYZ is given by $A = \frac{1}{2}xy \sin Z$.

Note that the angle Z must be the one between the sides of length x and y.

Example 6.3

Find the area of triangle PQR in Figure 6.14 correct to one decimal place.

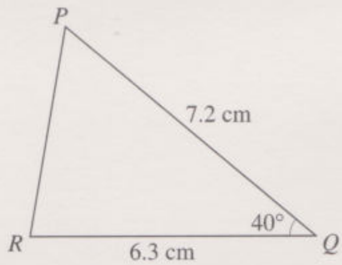


Figure 6.14

Solution

Since Q is the known angle, the area of the triangle is

$$\begin{aligned}\tfrac{1}{2} \times PQ \times QR \times \sin Q &= \tfrac{1}{2} \times 7.2 \times 6.3 \times \sin 40^\circ \\ &= 14.6\,\text{cm}^2 (\text{ to 1 d.p.}).\end{aligned}$$

Exercise 6.8

Find the area of triangle ABC correct to one decimal place.

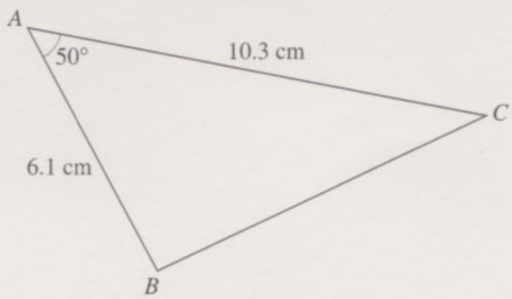


Figure 6.15

Module 7 Graphs and functions

7.1 Graphs

A **graph** is often used in mathematics to display the relationship between two quantities.

Distance and time graphs

In a **distance–time graph** the vertical axis represents the distance travelled from a fixed point and the horizontal axis represents the time taken.

Imagine that a journey of distance S is completed in time T (distance and time are both measured from 0). One way to represent this journey as a graph of distance against time is to assume that throughout the journey the speed of travel is constant, so the distance travelled is directly proportional to the time elapsed.

This assumption leads to a straight-line graph starting at the origin and finishing at the point with coordinates (S, T), as in Figure 7.1.

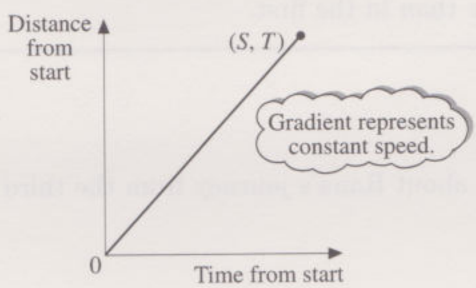


Figure 7.1

The ratio

$$\frac{\text{distance travelled}}{\text{time taken}},$$

which is the gradient of the line, gives the constant speed of travel.

Without the assumption of a constant speed of travel, the same graph can be used to represent *average* progress through the journey. In this case the gradient of the line gives the average speed, as indicated below.

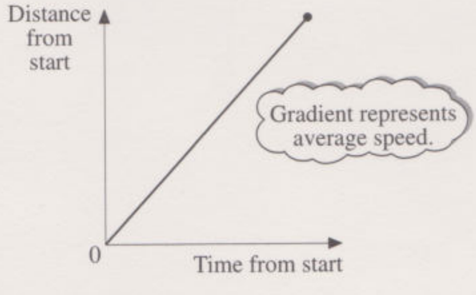


Figure 7.2

Many journeys however are better represented with parts of the graph showing different average speeds for different stages of the journey.

Example 7.1

The graph below represents a more complicated journey, taken by Rana. It is in four stages, each with a different average speed.

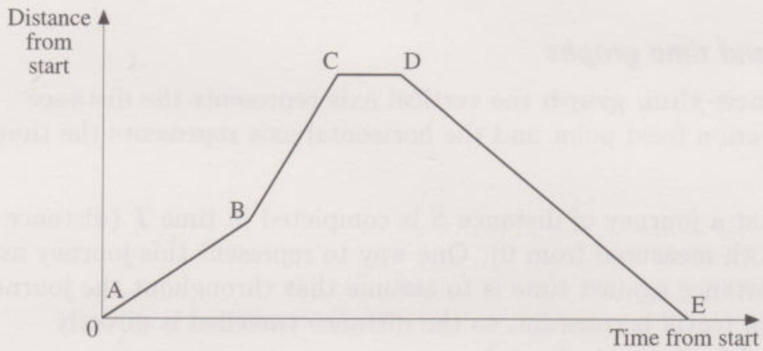


Figure 7.3

For the stage from B to C, the gradient of the graph is steeper than that for the stage from A to B. This shows that her average speed in this second stage is greater than in the first.

Exercise 7.1

What can you deduce about Rana's journey from the third and fourth sections of the graph?

Exercise 7.2

A cross-channel ferry usually takes 2 hours to make the 40 km crossing from England to France. Assuming that the ferry leaves at 12.00 and the speed of the ferry is constant during the crossing, draw a distance–time graph to represent the journey. From your graph find:

(a) when the ferry is 15 km and 35 km from England;

(b) where the ferry is at 12.15 and at 13.10;

(c) the gradient of the graph;

(d) the average speed of the ferry during the crossing.

Graphs of proportionality

The relationship exhibited in Figure 7.1 is one of **direct proportion**. Other such relationships are converting the units of a physical quantity to another unit (ounces to grams, for example) and currency conversion (US dollars to pounds sterling, for example).

The graph of any relationship in which one quantity y is directly proportional to another quantity x has the following form.

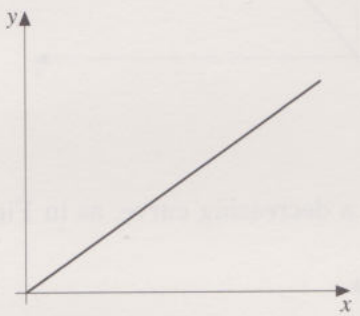


Figure 7.4

The equation of the graph is $y = kx$ where k is a constant. The value of k is the gradient of the line. We write 'y is proportional to x' as $y \propto x$.

Direct proportionality is just one example of the family of proportional relationships which have the general form:

$y = kx^n$, where k is a positive constant.

When $n = 1$, $y = kx$ and the graphs are of the form shown in Figure 7.4.

When $n > 1$, the form of the graph is as in Figure 7.5; it is an increasing curve through the origin.

In particular, when $n = 2$, $y \propto x^2$ and the equation of the graph is $y = kx^2$.

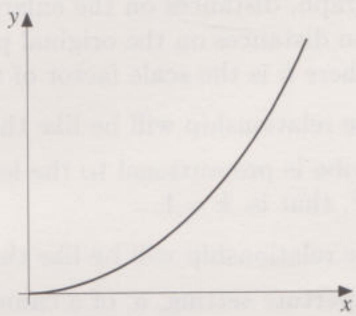


Figure 7.5

In particular, when $n = \frac{1}{2}$, $y \propto \sqrt{x}$ and the equation of the graph is $y = kx^{\frac{1}{2}}$. Remember that $\sqrt{x}$ is the positive square root of x.

When $0 < n < 1$, the form of the graph is as in Figure 7.6. It is also an increasing curve through the origin, but notice the difference in shape.

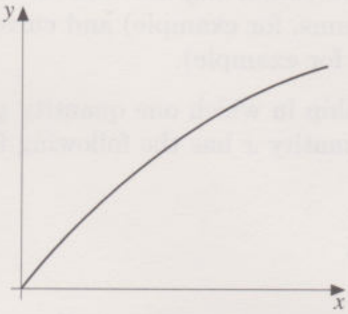


Figure 7.6

When $n = -1$, $y \propto 1/x$: y is **inversely proportional**, to x.

When $n < 0$, the graph is a decreasing curve, as in Figure 7.7.

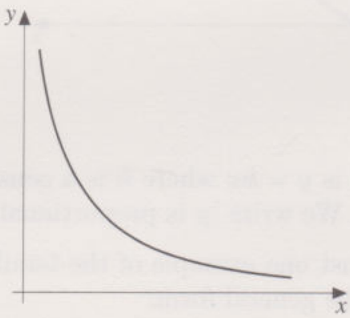


Figure 7.7

Example 7.2

(a) In an enlarged photograph, distances on the enlargement (y) are directly proportional to distances on the original photograph (x). That is, $y \propto x$ or $y = kx$, where k is the scale factor of the enlargement.

The graph showing the relationship will be like the one in Figure 7.4.

(b) The volume (y) of a cube is proportional to the length of its sides (x): $y \propto x^3$. In fact, $y = x^3$, that is, $k = 1$.

The graph showing the relationship will be like the one in Figure 7.5.

(c) Table 7.1 relates the aperture setting, a, of a camera to the distance in metres, d, from the photographic subject.

Table 7.1

d	1.00	1.40	2.00	2.75	5.50
a	22	16	11	8	4

Careful inspection shows that for each column in the table, except the second, $ad = 22$. (In this case $ad = 22.4$, so it is probable that one of the figures has been rounded.) That is, $a = 22/d$.

So a is inversely proportional to d ($a \propto 1/d$) and the graph of the relationship will be of type shown in Figure 7.7.

Exercise 7.3

For each of the following write down a formula which expresses the relationship between the first and second quantities, and say which kind of graph will display this.

(a) The area, A, of a circle and its radius, r.

(b) The radius of a circle, r, and its area, A.

(c) The average speed, s, in kilometres per hour and the time, t, in hours taken for a journey from London to Edinburgh, given that the distance between London and Edinburgh is 650 km.

(d) The number of US dollars, D, and the corresponding number of pounds sterling, P, if US\$1 = £0.66.

7.2 Functions

In many real-world situations, variables are related in more complicated ways than those indicated so far. To deal with such situations, the more general notion of a **function** is introduced.

Suppose that the relationship between two variables x and y, say, can be described by an equation of the form

Any two letters may be used in place of x and y.

$$y = \text{expression involving } x, \text{ but not } y.$$

Moreover, suppose that for each value of x substituted in the right-hand side just one value of y is obtained. Examples of such relationships are given by the equations

Such relationships are said to be 'functional relationships.'

$$y = x^2, \; y = 3x + 1 \text{ and } y = \sqrt{x}.$$

(Note that the relationship given by

$$y = \pm\sqrt{1 - x^2},$$

obtained from $x^2 + y^2 = 1$, is *not* such a relationship. The equation gives *two* values for each value of x except $x = 1$ and $x = -1$.)

In such relationships x is called the **independent variable** and y is called the **dependent variable**, and y is said to be **'a function of** x'. To signify this, we write

Any letter may be used in place of f (other than x and y in this context). Common choices are g and h.

$$y = f(x);$$

in the above three examples, $f(x) = x^2$, $f(x) = 3x + 1$ and $f(x) = \sqrt{x}$.

Specific functions are referred to as 'the function $f(x) = x^2$, the function $g(x) = 2x$', and so on.

The set of values x that can be substituted into a function is called the **domain** of the function. Since any real number can be substituted in the expression $3x^2 + 1$, the function $f(x) = 3x^2 + 1$ has the set of all real numbers as domain. The function $g(x) = \sqrt{x}$ has domain the set of all positive numbers and zero. To indicate their domains, we specify these functions as follows:

$$f(x) = 3x^2 + 1 \qquad (x \text{ is real})$$

and

$$g(x) = \sqrt{x} \qquad (x \geq 0).$$

For each input number, there is a single corresponding output number.

It is helpful to think of a function in terms of **input** values (from the domain), a **process** or **rule** (such as $3x^2 + 1$ or $\sqrt{x}$) and the corresponding **output** values. The set of all output numbers is often referred to as the **codomain**.

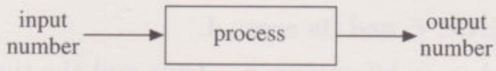


Figure 7.8

Exercise 7.4

For each of the following input numbers,

$-4, -1, 0, 1, 4$

find (where possible) the corresponding output number for each of the functions:

(a) $f(x) = 3x^2 + 1$;

(b) $g(x) = \sqrt{x}$.

Functions are used to model many physical situations. Here is a familiar example, in which the physical situation dictates the domain.

Example 7.3

The formula for the area A of a square with sides of length s is

$A = s^2$.

In function form this can be expressed as

$A = f(s), \quad \text{where } f(s) = s^2$.

Since the length of any square must be positive, the full description of the function is

$f(s) = s^2 \qquad (s > 0)$.

(In the absence of the modelling context, the function $g(x) = x^2$ has domain the set of all real numbers.)

Here is a summary.

Your calculator has keys to represent a number of mathematical functions. For each allowable input, your calculator returns the corresponding output.

A **function** is a relationship between two variables x and y, say, such that for each allowable value of x (the input) there is just one value of y (the output). The set of all allowable values of x is called the **domain** of the function, and the set of outputs is called the **codomain**. Applying the **process** or **rule** of the function to each input gives the corresponding output. The domain, codomain and process are the ingredients of a function.

Graph sketching

The **graph of a function** is the graph of the corresponding equation, for values in its domain. For example, the graph of the function

$$f(x) = x^2 \qquad (x > 0)$$

is the graph of $y = x^2$ for x positive.

It is often helpful to be able to *visualise* the general shape of the graph of a function expressed algebraically. It is not necessary to *plot* a graph in order to visualise its general shape: all that is necessary is to know the key points, such as where it crosses the axes or changes direction. A first step towards being able to visualise graphs is learning to *sketch* them.

Example 7.4

Sketch the graph of each of the following functions.

(a) $f(x) = -2x + 2$

(b) $f(x) = x^2 - 3 \qquad (x \geq 0)$

When a function has no domain given, it is assumed to be the set of all real numbers to which the rule can be applied.

Solution

(a) From a knowledge of linear equations, $y = -2x + 2$ is the equation of a straight line with a negative gradient (-2) and y-intercept of $+2$. So the line would look like Figure 7.9.

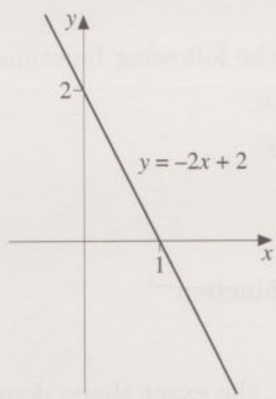


Figure 7.9

(b) $y = x^2 - 3$ is a quadratic like $y = x^2$. Its graph is the same as that of $y = x^2$, but moved (translated) -3 in the y-direction, i.e. downwards. So it would look like Figure 7.10.

But the function was limited by the domain being the set of non-negative real numbers, so the sketch required is only values corresponding to non-negative values of x as in Figure 7.11.

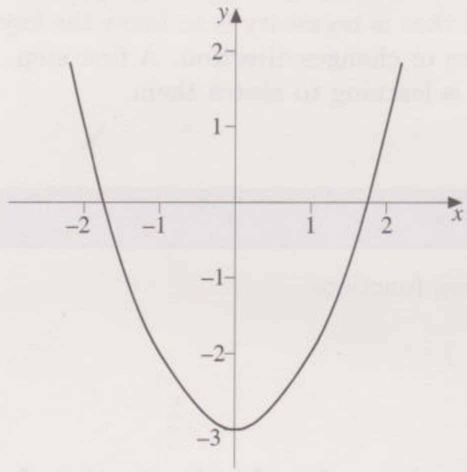


Figure 7.10

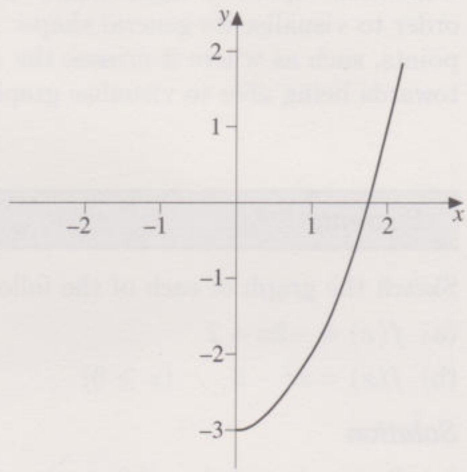


Figure 7.11

Exercise 7.5

Sketch the graph of each of the following functions.

(a) $f(x) = 2x + 3 \qquad (x \geq 0)$

(b) $f(x) = x^2 + 2 \qquad (x \leq 0)$

Quadratic functions

The graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

is a parabola (see Module 5), the exact shape depending on the values of a, b and c. The following example considers three special cases:

(a) $b = 0$, $c = 0$; (b) $b = 0$, $c \neq 0$; (c) $c = 0$, $b \neq 0$.

Example 7.5

(a) $b = 0$, $c = 0$.

Figure 7.12 shows the graph of $y = ax^2$ for various values of a. Notice that as a increases from 1, the parabola becomes 'narrower', and as a decreases from 1 towards 0, the parabola becomes 'wider'. Also note that if a is negative, the parabola is the reflection in the x-axis of the parabola for the corresponding positive value of a.

If $a > 0$, the parabola has a lowest point, which corresponds to the **minimum value** of the function. If $a < 0$, the parabola has a highest point, which corresponds to the **maximum value** of the function.

The point where these values occur is the **vertex of the parabola**. The graph touches the x-axis at this point only.

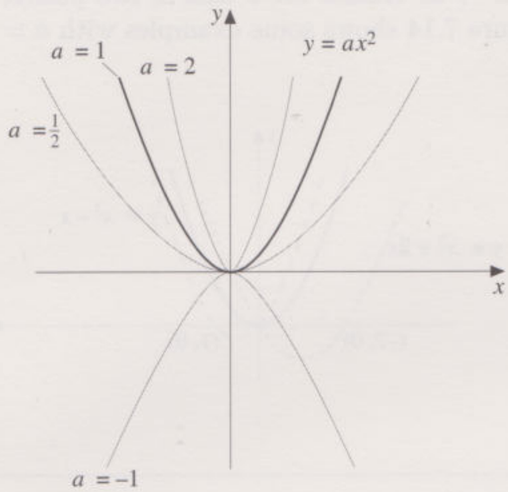


Figure 7.12

(b) $b = 0$, $c \neq 0$.

Compared to the graph of $y = ax^2$, the graph of $y = ax^2 + c$ is moved up ($c > 0$) or down ($c < 0$) the y-axis. Figure 7.13 shows some examples with $a = 1$.

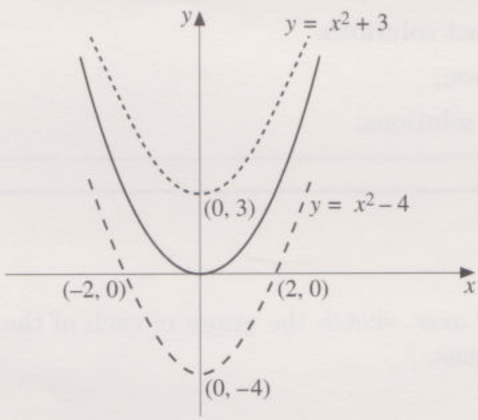


Figure 7.13

Notice that the graph of $y = x^2 - 4$ cuts the x-axis at -2 and 2. These are the solutions of the equation $x^2 - 4 = 0$. On the other hand, $y = x^2 + 3$ does not cut the x-axis – that is, $x^2 + 3 = 0$ has no real solutions.

(c) $c = 0$, $b \neq 0$.

Compared to the graph of $y = ax^2$, the graph of $y = ax^2 + bx$ is moved down and to the right (if $b < 0$) or down and to the left (if $b > 0$). The graph of $y = ax^2 + bx$ crosses the x-axis in two places, one of which is the origin. Figure 7.14 shows some examples with $a = 1$.

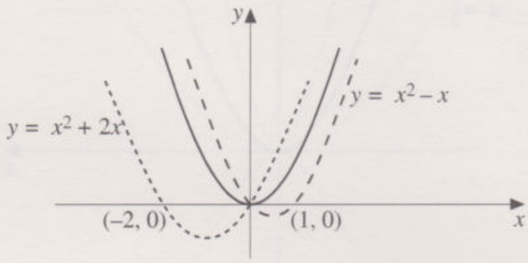


Figure 7.14

> Depending on the values of a, b and c, the graph of the function $f(x) = ax^2 + bx + c$ behaves in one of the following ways.
>
> ◇ It crosses the x-axis at two distinct points.
>
> ◇ It touches the x-axis at one point.
>
> ◇ It does not cross or touch the x-axis.
>
> In terms of the equation $ax^2 + bx + c = 0$, these cases correspond respectively to:
>
> ◇ two distinct solutions;
>
> ◇ one solution;
>
> ◇ no (real) solutions.

Exercise 7.6

(a) Using one set of axes, sketch the graph of each of the following quadratic functions.

(i) $f(x) = -x^2$

(ii) $f(x) = -2x^2$

(iii) $f(x) = -2x^2 + 2$

(b) Using another set of axes, sketch the graph of each of the following quadratic functions.

(i) $f(x) = x^2$

(ii) $f(x) = x^2 + 3x$

(iii) $f(x) = x^2 - 3x$

Trigonometric functions

There is a trigonometric function corresponding to each of the trigonometric ratios. The aim in this section is to show how these functions, whose input values are real numbers, are defined.

In Module 6 the trigonometrical ratios were defined using right-angled triangles, for angles between 0 and $\pi/2$, i.e. acute angles. We next define the ratios for angles greater than $\pi/2$ and for angles less than 0. We have purposely used radian measure for angles here since it is that measure which leads to the definitions of the trigonometric functions.

Figure 7.15 shows a circle with unit radius, centre $O(0,0)$ and a point $C(x, y)$. We define θ radians to be an angle measured anticlockwise from the positive x-axis. (Working with a circle of unit radius simplifies the following text since the hypotenuse in triangle OCX has length 1.)

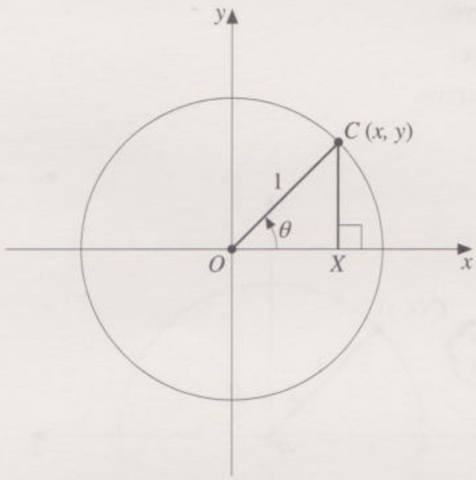


Figure 7.15

Example 7.6

In Figure 7.15, show that $x = \cos\theta$.

Solution

The x-coordinate of C is equal to the length OX.

From the right-angled triangle OCX, $\cos\theta = \dfrac{OX}{1} = OX$; therefore the x-coordinate of C is $x = \cos\theta$.

Exercise 7.7

(a) In Figure 7.15, show that $y = \sin\theta$.

(b) Write an expression for $\tan\theta$ in terms of x and y.

In Figure 7.15, C has two positive coordinates and the corresponding values of the three basic trigonometric ratios are

$$\sin\theta = y, \ \cos\theta = x, \ \tan\theta = \frac{y}{x}.$$

These relationships provide an alternative definition of the three trigonometric ratios, for θ between 0 and $\pi/2$ (radians). They are also used to define these ratios for any value of θ from 0 to 2π (radians).

The signs of x and y vary as C moves round the circle. For example, if C is in the second quadrant, so that θ is between $\pi/2$ and π (radians), then x is negative and y is positive. Thus

$\cos\theta = x$ is negative;

$\sin\theta = y$ is positive;

$\tan\theta = y/x$ is negative.

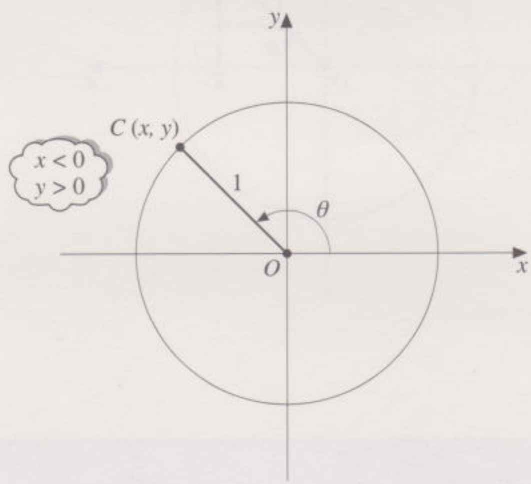


Figure 7.16

tan θ is not defined when $x = 0$, that is for $\theta = \pi/2$ and $\theta = 3\pi/2$.

> For $0 \leq \theta < 2\pi$ the trigonometric ratiosare defined by
>
> $$\cos\theta = x, \ \sin\theta = y, \ \tan\theta = y/x,$$
>
> where (x, y) are the coordinates of a point C anywhere on a circle of radius 1 and centre the origin, O, and θ radians is the angle between OC and the positive x-axis (measured anticlockwise).

Example 7.7

In Figure 7.17, the point P has coordinates $(-\sqrt{3}/2, 1/2)$, and triangle OPX is right-angled.

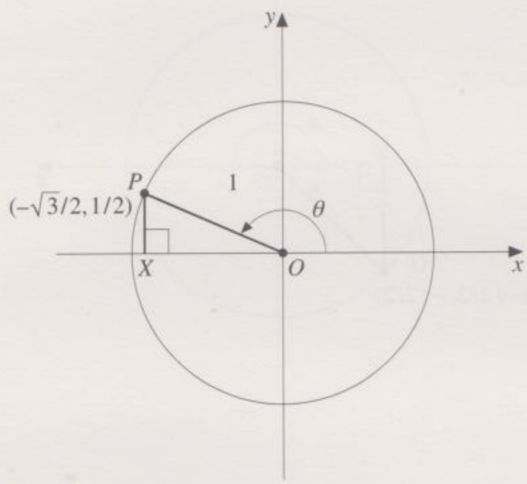


Figure 7.17

In triangle OPX, $\cos \angle POX = \sqrt{3}/2$, and so $\angle POX = \pi/6$ (see Example 6.2).

$\frac{1}{2}/\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$.

Hence

$$\theta = \pi - \frac{\pi}{6} = \frac{5}{6}\pi.$$

(a) Use the definitions to write down the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

(b) Check the values by using a calculator.

Solution

(a) Using the coordinates of P, we have

$$\sin\theta = \frac{1}{2}, \ \cos\theta = -\frac{\sqrt{3}}{2}$$

and

$$\tan\theta = \frac{1}{2}\Big/\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{2} \times \left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

(b) Using a calculator,

$$\sin\theta = \sin\frac{5}{6}\pi = 0.5000 \text{ (to 4 d.p.)};$$

$$\cos\theta = \cos\frac{5}{6}\pi = -0.8660 \text{ (to 4 d.p.)};$$

$$\tan\theta = \tan\frac{5}{6}\pi = -0.5774 \text{ (to 4 d.p.)}.$$

These values are the decimal equivalents of $\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$ and $-\frac{\sqrt{3}}{3}$.

Exercise 7.8

(a) In Figure 7.18, the point Q has coordinates $(-\sqrt{2}/2,\ -\sqrt{2}/2)$, and α radians is the angle between OQ and the positive x-axis.

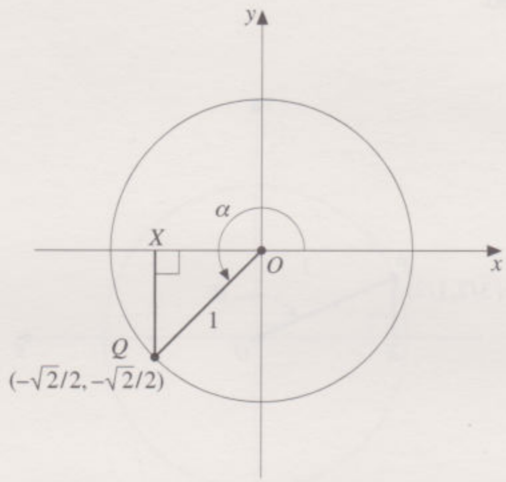


Figure 7.18

(i) What is the value of α?

(ii) Use the coordinates of Q to write down the values of $\cos\alpha$, $\sin\alpha$ and $\tan\alpha$.

(iii) Check the values of $\cos\alpha$, $\sin\alpha$ and $\tan\alpha$ by using a calculator.

(b) In Figure 7.19, the point R has coordinates $(1/2, -\sqrt{3}/2)$ and β radians is the angle between OR and the positive x-axis.

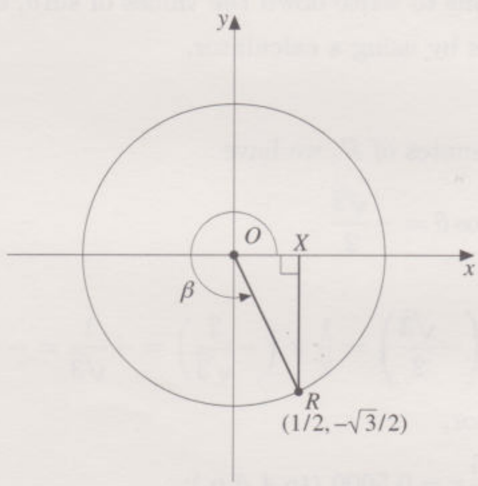


Figure 7.19

(i) What is the value of β?

(ii) Use the coordinates of R to write down the value of $\cos\beta$, $\sin\beta$ and $\tan\beta$.

(iii) Check the values of $\cos\beta$, $\sin\beta$ and $\tan\beta$ by using a calculator.

From the results of the above example and exercise notice that the sin, cos and tan of angles greater than $\pi/2$ change sign dependent on the quadrant. Figure 7.20 summarises this information on signs.

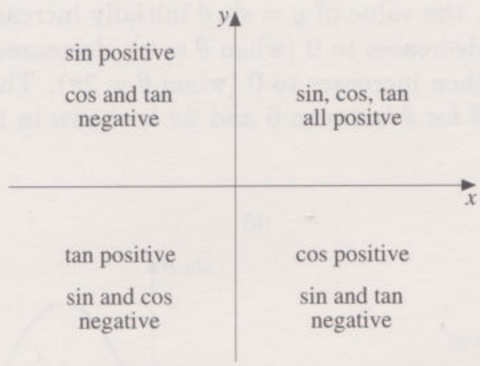
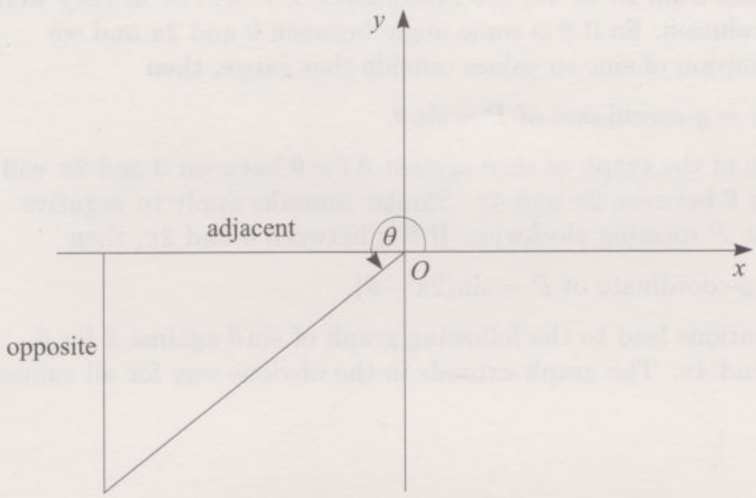


Figure 7.20

In general you will find that sin, cos and tan are positive for angles in the first quadrant (0° to 90° or 0 to $\frac{\pi}{2}$ radians). In the other three quadrants one is positive and the other two are negative. The signs of sin, cos and tan for angles in each quadrant can be determined from the directions of the opposite and adjacent sides of the associated right-angled triangle in that quadrant. For example, in the third quadrant (180° to 270° or π to $\frac{3}{2}\pi$ radians) both the opposite and adjacent sides are in negative directions (down and to the left respectively).



So the sine and the cosine are both negative but the tangent is positive. Many people remember the information about signs using the following picture (positive ratios indicated) with one of the mnemonics 'All sing the chorus' or 'CAST'.

Look at Figure 7.21(a). As $P(x, y)$ move round the unit circle, i.e. as θ increases from 0 to 2π, the value of $y = \sin\theta$ initially increases from 0 to 1 (when $\theta = \pi/2$), then decreases to 0 (when $\theta = \pi$), decreases further to -1 (when $\theta = 3\pi/2$) and then increases to 0 (when $\theta = 2\pi$). The resulting graph of $\sin\theta$ against θ for θ between 0 and 2π is shown in Figure 7.21(b).

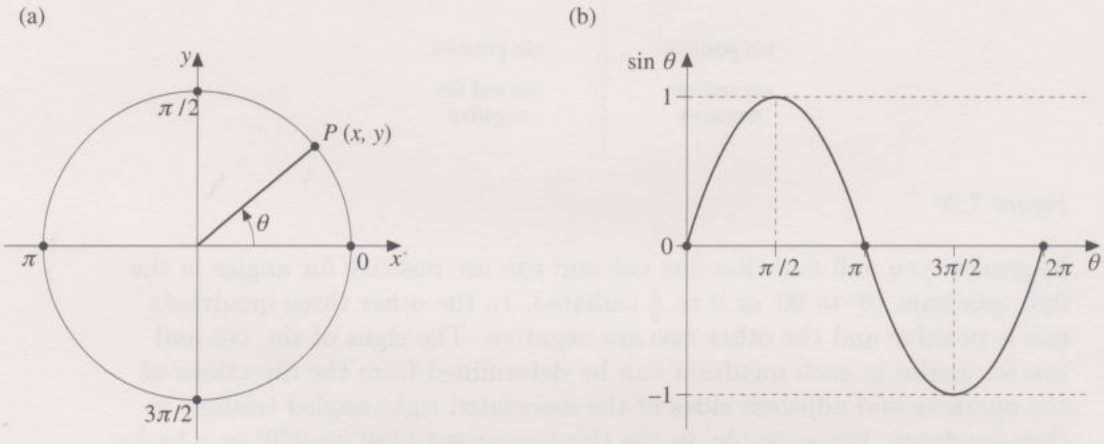


Figure 7.21

Imagine $P(x, y)$ making a second revolution of the circle. As the angle of rotation increases from 2π to 4π, the coordinates of P will be as they were on the first revolution. So if θ is some angle between 0 and 2π and we extend the definition of sine to values outside that range, then

$$\sin(\theta + 2\pi) = y\text{-coordinate of } P = \sin\theta.$$

Thus the shape of the graph of $\sin\theta$ against θ for θ between 0 and 2π will be repeated for θ between 2π and 4π. Similar remarks apply to negative angles. Imagine P rotating clockwise. If θ is between 0 and 2π, then

$$\sin(-\theta) = y\text{-coordinate of } P = \sin(2\pi - \theta).$$

These considerations lead to the following graph of $\sin\theta$ against θ for θ between -4π and 4π. The graph extends in the obvious way for all values of θ.

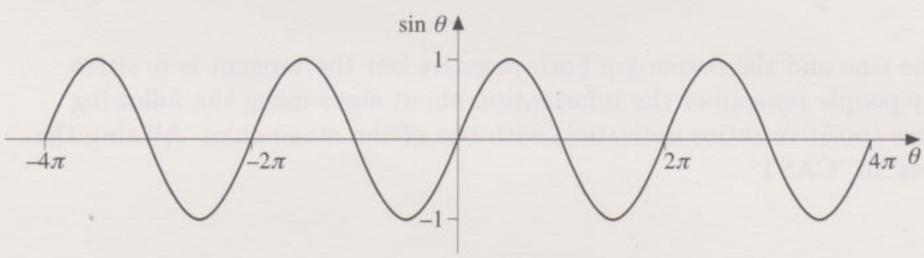


Figure 7.22

The sine function $f(x) = \sin x$

In the above discussion, involving angles represented by θ radians, θ is any real number and $\sin\theta$ is the number, between -1 and 1, corresponding to θ. Thus we have the ingredients for a function. To emphasise this, we write

$$f(x) = \sin x \qquad (x \text{ is real})$$

where we have used the customary x for input values. The graph of the function $f(x) = \sin x$ (Figure 7.23) is the same as Figure 7.22, but with θ replaced by x and $\sin\theta$ replaced by y.

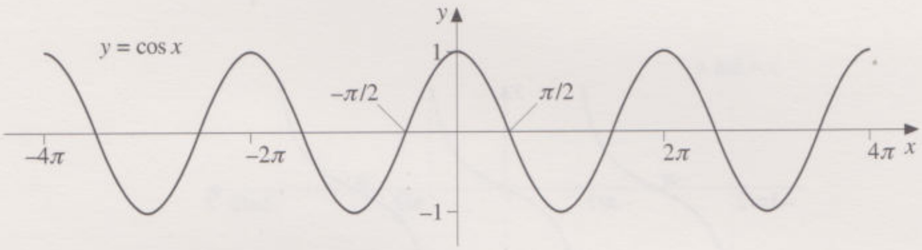


Figure 7.23

If this graph were extended in either direction more repetitions of the basic shape for x between 0 and 2π would be obtained.

Such a function is said to be **periodic**. The function $f(x) = \sin x$ has **period** 2π.

The cosine function $f(x) = \cos x$

The graph of the function $f(x) = \cos x$ is shown in Figure 7.24.

The argument to justify the shape of the graph is similar to that used for the sine function.

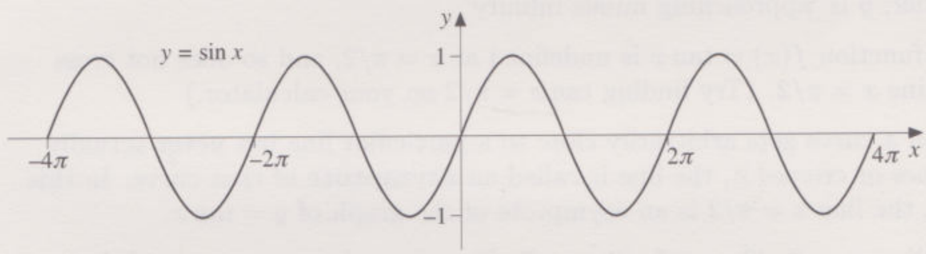


Figure 7.24

Like the sine function, the cosine function has domain the real numbers and is periodic, with period 2π. Its values lie between the maximum value 1 and minimum value -1, both of which occur infinitely many times.

The graph of the sine and cosine functions have the same shape: shifting the former to the left by $\pi/2$ gives the latter. This is summarised in the relationship

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x, \quad \text{for all } x.$$

The tangent function

The graph of the function $f(x) = \tan x$ is shown below.

It comprises infinitely many disjoint parts, each with the same shape.

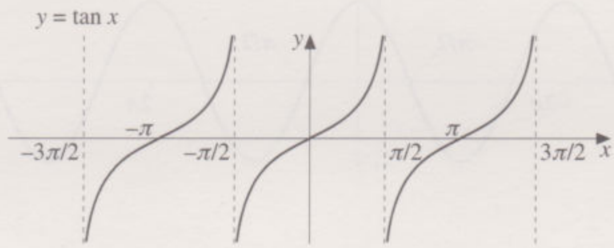


Figure 7.25

The tangent function is periodic but with period π, but in other respects its graph looks very different from those for the sine and cosine functions. It has no maximum or minimum values.

Looking at values near to $\pi/2$, it can be seen that when x is less than $\pi/2$, but is getting closer and closer to $\pi/2$ from the left, the corresponding values of y get larger and larger; y is 'approaching infinity'.

On the other hand when x is bigger than $\pi/2$ but getting closer and closer to $\pi/2$ from the right, the corresponding values of y get smaller and smaller; y is 'approaching minus infinity'.

The function $f(x) = \tan x$ is undefined at $x = \pi/2$, and so does not cross the line $x = \pi/2$. (Try finding $\tan x = \pi/2$ on your calculator.)

When a curve gets arbitrarily close to a particular line but never actually reaches or crosses it, the line is called an **asymptote** of that curve. In this case, the line $x = \pi/2$ is an asymptote of the graph of $y = \tan x$.

The lines $x = 3\pi/2$, $x = 5\pi/2$, $x = 7\pi/2$, ... are also asymptotes of the graph of $y = \tan x$, as are the lines $x = -\pi/2$, $x = -3\pi/2$, $x = -5\pi/2$, and so on.

Exponential and logarithmic functions

The exponential function $f(x) = e^x$

The basic exponential function is

$$f(x) = e^x \qquad (x \text{ is real}),$$

where e is the base of natural logarithms (Module 1). Its codomain is the set of positive real numbers. Its graph is shown below.

An alternative way of writing this function is

$$f(x) = \exp(x).$$

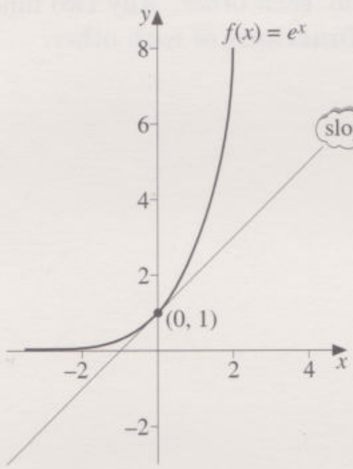


Figure 7.26

The graph passes through the point $(0, 1)$, is always positive and approaches the negative x-axis asymptotically. Note also that the gradient of the graph at $x = 0$ is 1.

The natural logarithm function

The graph of the natural logarithm function

$$g(x) = \ln x \qquad (x > 0)$$

is shown below.

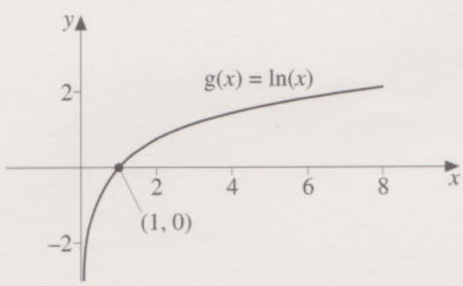


Figure 7.27

Notice that the natural logarithm function is defined only for positive values of x, and that its graph is asymptotic to the negative y-axis. The codomain of this function is the set of real numbers.

There is a close connection between the natural logarithm function and the exponential function.

Exercise 7.9

(a) (i) For $x = 1$, use your calculator to evaluate e^x. Then find the natural logarithm of the result.

(ii) Repeat part (i) for $x = 2$.

(b) (i) For $x = 1$, use your calculator to evaluate $\ln x$. Then evaluate $e^{\ln 1}$.

(ii) Repeat part (i) for $x = 2$.

(c) What do you notice in parts (a) and (b)?

This exercise illustrates the fact that the natural logarithm function and the exponential function 'undo' each other. Any two functions with this property are called **inverse functions** of each other.

Module 8 Geometry

Geometry is the study of one-, two-, and three-dimensional objects in space and has generated a precise vocabulary to describe some of the properties of objects. One emphasis in this module is on recognising geometrical properties of objects and using the associated algebraic terminology accurately. The other is on achieving familiarity with some common formulas associated with the 'size' of objects (length, area and volume).

8.1 Properties of plane figures

A **plane figure** is any figure which can be drawn in the plane. It is therefore two-dimensional.

Polygons

A plane closed figure whose sides are (straight) line segments is called a **polygon**. Many of the polygons with small numbers of sides have special names: for example, triangle, quadrilateral, **pentagon** (5 sides), **hexagon** (6 sides).

In general, a polygon with n sides (and hence n angles) is referred to as an **n-gon**.

Recall that the angle sum of a triangle is 180° and that of a quadrilateral is 360°. These facts can be generalised for polygons with more than four sides.

Example 8.1

$ABCDE$ is a pentagon. It is divided into three triangles by the line segments AC and AD – see Figure 8.1. (AC and AD are the **diagonals** of the pentagon from A.)

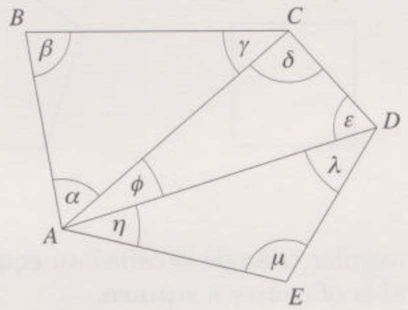


Figure 8.1

The sum of the angles of the triangle ABC is $\alpha + \beta + \gamma = 180°$.

The sum of the angles of the triangle ACD is $\delta + \varepsilon + \phi = 180°$.

The sum of the angles of the triangle ADE is $\lambda + \mu + \eta = 180°$.

δ (delta), λ (lambda), ε (epsilon), μ (mu), ϕ (phi), η (eta).

The sum of the five angles of the pentagon is

$$(\eta + \phi + \alpha) + \beta + (\gamma + \delta) + (\varepsilon + \lambda) + \mu$$
$$= (\alpha + \beta + \gamma) + (\delta + \varepsilon + \phi) + (\lambda + \mu + \eta)$$
$$= 3 \times 180^\circ = 540^\circ.$$

The following rearrangement of the sums of the angles of a triangle, quadrilateral and pentagon enable us to see the general formula for the sum of the angles of an n-gon.

Angle sum of a triangle $= 180^\circ = (3 - 2) \times 180^\circ$;

angle sum of a quadrilateral $= 360^\circ = (4 - 2) \times 180^\circ$;

angle sum of a pentagon $= 540^\circ = (5 - 2) \times 180^\circ$.

> In general, the angle sum of an n-gon is $(n - 2) \times 180^\circ$.

Exercise 8.1

Use the above result to calculate the angle sums of each of the following plane figures.

(a) a **heptagon** (or 7-gon)

(b) a **nonagon** (or 9-gon)

(c) a **decagon** (or 10-gon)

Regularity

> A polygon is said to be **regular** if all its sides are equal and all its angles are equal.

Figure 8.2 shows a regular triangle, a regular quadrilateral and a regular pentagon.

Figure 8.2

As stated in Module 3, a regular triangle is called an equilateral triangle, and a regular quadrilateral is of course a **square**.

However, triangles which are not regular also have commonly-used, special names (again, see Module 3), and the same is true for some quadrilaterals, though not in general for polygons with more than four sides. Some special cases of quadrilaterals are shown in Figure 8.3.

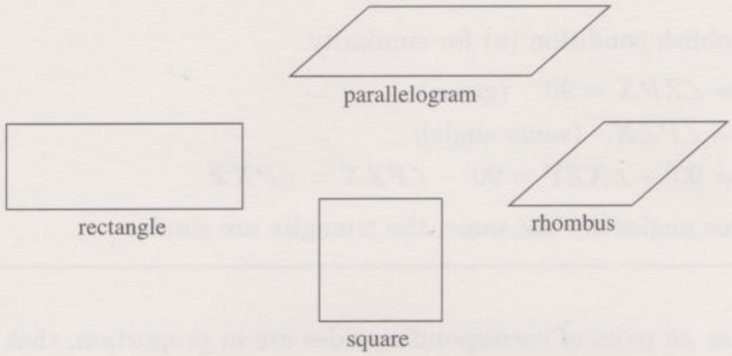


Figure 8.3

A **parallelogram** has opposite sides equal and opposite angles equal.

A parallelogram in which all angles are equal is called a **rectangle**.

Since the angle sum of a quadrilateral is 360°, each of the angles of a rectangle is 90°.

A parallelogram in which all sides are equal is called a **rhombus**.

A rectangle which is also a rhombus is a square.

Similarity

Any two figures are said to be **similar** if they are the same shape, though they may well be of different sizes. Essentially, one figure is a **scaling** of the other.

Similar triangles occurred in Module 6.

Two polygons are similar if each angle in one of them is equal to its counterpart in the other and the length of each line in one is the same multiple of its counterpart in the other.

In the case of triangles (or 3-gons) this means that two triangles are similar if any *one* of the following conditions can be shown to be true:

(a) all the angles of one triangle are the same as the angles of the other;

(b) all pairs of corresponding sides are in the same proportion;

(c) two pairs of corresponding sides are in the same proportion and the included angle is the same.

Example 8.2

In the triangle XYZ, $\angle YXZ$ is a right angle.

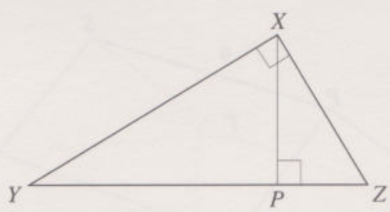


Figure 8.4

A line is drawn from X meeting YZ at P, so that $\angle XPZ$ is also a right angle. Show that triangles XYZ and PXZ are similar.

Solution

We shall establish condition (a) for similarity.

$\angle YXZ = \angle ZPX = 90^\circ$ (given);
$\angle XZY = \angle PZX$ (same angle);
$\angle XYZ = 90^\circ - \angle XZY = 90^\circ - \angle PZX = \angle PXZ.$

Since all three angles are the same, the triangles are similar.

It follows that all pairs of corresponding sides are in proportion, that is,

$$\frac{YX}{XP} = \frac{XZ}{PZ} = \frac{YZ}{ZX} = k;$$

so $YX = kXP$, $XZ = kPZ$ and $YZ = kZX$.

Exercise 8.2

(a) Identify a third triangle in Figure 8.4 which is similar to triangles XYZ and PXZ.

(b) Write down $\sin \angle XZY$ and $\sin \angle PZX$, and show that these two expressions are equal.

Congruence

Two figures which are the same shape *and* the same size are said to be **congruent**.

8.2 Areas and volumes of solids

A **solid** is a term used in geometry to denote a three-dimensional object.

Prisms

All the prisms we consider are right prisms, so we shall refer to them as 'prisms', from here on.

A right **prism** is a solid such that wherever a cut is made through it in a direction at right angles to a direction through it, all the cut surfaces are congruent. The common area of the cut surfaces is called the **cross-sectional area** of the prism. Each prism has two end faces which are congruent and parallel to the cut surfaces. The distance between the end faces is called the **length** of the prism.

Figure 8.5 shows the triangular prism $PQRSTU$.

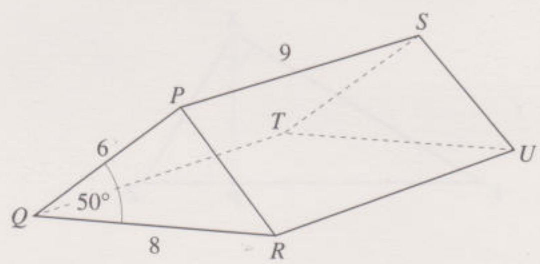


Figure 8.5

The **faces** PQR and STU at the ends of the prism are congruent triangles, and the three other faces are rectangles. All the cut surfaces at right angles to the direction defined by PS (for example) are congruent.

The line where two faces of a solid meet is called an **edge**. For example, PQ is the edge of the prism where the faces PQR and $PQTS$ meet.

The point at which three (or more) edges of a solid meet is called a **vertex**. For example, P is the vertex where the edges PQ, PR and PS meet.

The plural of vertex is vertices.

A **cube**, which has six square faces, can be thought of as a **square prism**: see Figure 8.6.

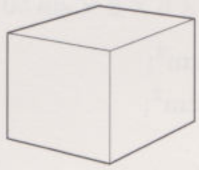


Figure 8.6

A **cuboid**, which has three pairs of matching rectangular faces, is a **rectangular prism**: see Figure 8.7.

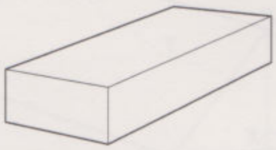


Figure 8.7

A **cylinder**, which has two plane circular faces and a curved face that can be opened out to form a rectangle, is a **circular prism**: see Figure 8.8.

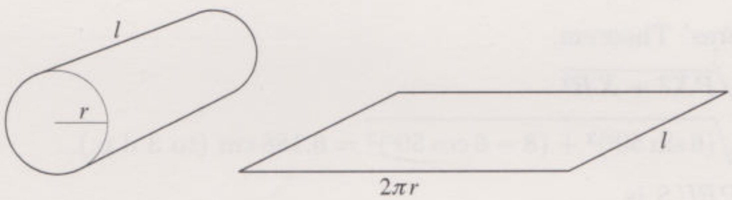


Figure 8.8

Surface area of prisms

The **surface area** of a prism is equal to the sum of the areas of all its faces.

Surface area of a sphere

The surface area of a sphere of radius r is $4\pi r^2$.

Example 8.3

Find the surface area of the triangular prism in Figure 8.5.

Solution

We have

$$\text{area } PQR = \text{area } STU = \frac{1}{2} \times 6 \times 8 \times \sin 50^\circ = 18.39\,\text{cm}^2 \text{ (to 2 d.p.)};$$
$$\text{area of } PQTS = 6 \times 9 = 54\,\text{cm}^2;$$
$$\text{area of } TQRU = 8 \times 9 = 72\,\text{cm}^2;$$
$$\text{area of } PRUS = PR \times 9.$$

The length of PR is not known but can be calculated if a perpendicular is drawn from P meeting QR at X.

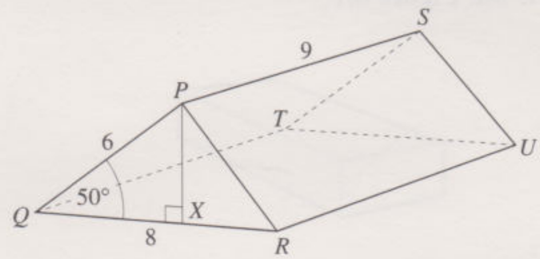


Figure 8.9

From triangle PQX, $PX = 6 \sin 50^\circ$ and $QX = 6 \cos 50^\circ$. Hence

$$XR = 8 - QX = 8 - 6\cos 50^\circ.$$

By Pythagoras' Theorem,

$$PR = \sqrt{PX^2 + XR^2}$$
$$= \sqrt{(6\sin 50^\circ)^2 + (8 - 6\cos 50^\circ)^2} = 6.188\,\text{cm (to 3 d.p.)}.$$

So area of $PRUS$ is

$$9 \times 6.188 = 55.69\,\text{cm}^2 \text{ (to 2 d.p.)}$$

and the surface area of the prism is

$$2(18.39) + 54 + 72 + 55.69 = 218.47\,\text{cm}^2 \text{ (to 2 d.p.)}.$$

Exercise 8.3

Find the surface area S of each of the following solids. (Draw a diagram for each to help you.)

(a) A cuboid which measures 6 cm by 7 cm by 10 cm.

(b) A cylinder of radius 10 cm and length 40 cm, giving your answer correct to two decimal places.

(c) A sphere of radius 4 cm.

(d) A sphere of radius 3 cm.

Volume of prisms

> The **volume** of a prism is equal to its cross-sectional area multiplied by its length.

Volume of a sphere

> The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

Example 8.4

Find the volume of the triangular prism in Figure 8.9 to one decimal place.

Solution

$$\begin{aligned}\text{Cross-sectional area of the prism} &= \text{area of triangle } PQR \\ &= 18.39\,\text{cm}^2 \text{ (to 2 d.p.)}.\end{aligned}$$

The length of the prism $= 9\,\text{cm}$.

Therefore, the volume of the prism is

$$18.39 \times 9 = 165.5\,\text{cm}^3 \text{ (to 1 d.p.)}.$$

Exercise 8.4

Find the volume V of each of the following solids.

(a) A cuboid which measures 6 cm by 7 cm by 10 cm.

(b) A cylinder of radius 10 cm and length 40 cm, giving your answer correct to two decimal places.

(c) A sphere of radius 4 cm.

Solutions to Exercises

Solution 1.1

(a) (i) 10 000 (ii) 100 000 (iii) 81

(b) (i) 7776 (ii) 341.8801 (iii) −357.911
(iv) −2 097 152 (v) 20 736

Solution 1.2

(a) (i) 1.427×10^9 (ii) 8.075×10^3
(iii) 3.27×10^{-3} (iv) 5.672×10^{-1}
(v) 4.007×10^{-7}

(b) (i) 329 800 (ii) 76.54 (iii) 0.001 098
(iv) 0.000 000 000 34

Solution 1.3

(a) (i) $3 + (5 \times 2) = 13$

(ii) $(10^3) \times 3 = 3000$

(iii) $\dfrac{(15+5)}{(3+7)} = \dfrac{20}{10} = 2$

At first sight this looks like an exception to 'division before addition'; but the long fraction bar is in effect 'all divided by', which implies brackets.

(iv) $6 - 4 + 2 = 2$ (no brackets necessary, the order does not matter in this case). Putting in brackets does change things: $(6-4)+2 = 4$; $6-(4+2) = 0$.

(v) $(2^2) + (3 \times (10^2)) = 4 + (3 \times 100) = 304$.

(b) Any differences will depend on *your* calculator.

Solution 1.4

(a) $441.7 \times 5.2 \simeq 400 \times 5 = 2000$.
Calculator gives 2296.84.

(b) $53.4 \times 70.9 \div 22.2 \simeq 50 \times 70 \div 20$
$= \dfrac{3500}{20} = \dfrac{350}{2} = 175.$
Calculator gives 170.543 243 2.

(c) $217.5 + 60.3 \times 17.7 \simeq 200 + 60 \times 20$
$= 200 + 1\,200 = 1400.$
Calculator gives 1284.81.

(d) $(1285 - 329) \times 0.023 \simeq (1300 - 300) \times 0.02$
$= 1000 \times 0.02 = 20.$
Calculator gives 21.988.

Solution 1.5

(a) (i) 2.1 (ii) 2.142 (iii) 2.1416
(iv) 2.14160

(b) (i) 63 000 (ii) 0.04 (iii) 0.0401

(c) (i) 23.0 (ii) 10 000 (iii) 6100
(iv) 17.0

Solution 1.6

(a) The factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18 and 36.

(b) $36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$

Solution 1.7

(a) The prime factors are as follows.

(i) $8 = 2 \times 2 \times 2 = 2^3$

(ii) $16 = 2 \times 2 \times 2 \times 2 = 2^4$

(iii) $32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$

(b) The HCF of 8, 16 and 32 is $2^3 = 8$.

Solution 1.8

(a) $28 = 2 \times 2 \times 7 = 2^2 \times 7$
$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2$

So the LCM of 28 and 36 is $2^2 \times 3^2 \times 7 = 252$.

(b) $7 = 7$
$10 = 2 \times 5$
$14 = 2 \times 7$

So the LCM of 7, 10 and 14 is $2 \times 5 \times 7 = 70$.

Solution 1.9

(a) 0.2 (b) $0.3333\ldots = 0.\dot{3}$ (c) 3.75
(d) 1.125 (e) $3.\dot{4}2857\dot{1}$

Solution 1.10

(a) (i) $\frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \frac{5}{6}$

(ii) $1\frac{3}{4} + \frac{3}{8} = \frac{7}{4} + \frac{3}{8} = \frac{14}{8} + \frac{3}{8} = \frac{17}{8} = 2\frac{1}{8}$

(iii) $\frac{18}{25} - \frac{2}{5} = \frac{18}{25} - \frac{10}{25} = \frac{8}{25}$

(iv) $4\frac{2}{7} - 1\frac{3}{5} = \frac{30}{7} - \frac{8}{5} = \frac{150}{35} - \frac{56}{35} = \frac{94}{35} = 2\frac{24}{35}$

(b) (i) $\frac{4}{5} \times \frac{2}{7} = \frac{8}{35}$

(ii) $\frac{4}{5} \div \frac{2}{7} = \frac{4}{5} \times \frac{7}{2} = \frac{28}{10} = \frac{14}{5} = 2\frac{4}{5}$

(iii) $1\frac{2}{25} \times 3\frac{8}{9} = \frac{27}{25} \times \frac{35}{9} = \frac{3}{5} \times \frac{7}{1} = \frac{21}{5} = 4\frac{1}{5}$

(iv) $4\frac{2}{7} \times 1\frac{2}{3} = \frac{30}{7} \times \frac{5}{3} = \frac{10}{7} \times \frac{5}{1} = \frac{50}{7} = 7\frac{1}{7}$

(v) $1\frac{2}{3} \div 1\frac{1}{14} = \frac{5}{3} \div \frac{15}{14} = \frac{5}{3} \times \frac{14}{15} = \frac{1}{3} \times \frac{14}{3}$
$= \frac{14}{9} = 1\frac{5}{9}$

In parts (iii)–(v), some use has been made of cross cancelling in products of fractions. For example, in $\frac{27}{25} \times \frac{35}{9}$, 9 is divided into 27 and 9, and 5 is divided into 25 and 35, to give $\frac{3}{5} \times \frac{7}{1}$.

Solution 1.11

(a) $\frac{1}{8} = 0.125$

(b) $-\frac{1}{10} = -0.1$

(c) $\frac{5}{1} = 5$

(d) $\frac{1}{25} = 0.04$

Solution 1.12

(a) 20% (b) 33.33% (to 2 d.p.) (c) 375%

(d) 112.5% (e) 342.86% (to 2 d.p.)

Solution 1.13

(a) 100 g (b) 10 g (c) 260 g (d) £26.25

(e) £100

Solution 1.14

(a) $(32.50 \div 50) \times 100\% = 65\%$

(b) $(23 \div 115) \times 100\% = 20\%$

(c) $(55 \div 50) \times 100\% = 110\%$

Solution 1.15

Original price (£)	Percentage increase	Multiplier	New price (£)
100	30%	1.3 (130/100)	130
45	3%	1.03 (103/100)	46.35
230	15%	**1.15**	**264.50**
120	150%	**1.5**	**180.00**
36	20%	1.2	43.20
76	17.5%	1.175	**89.30**
25.53	17.5%	**1.175**	30

Solution 1.16

Original price (£)	Percentage decrease	Multiplier	New price (£)
100	20%	0.8	80
200	25%	**0.75**	**150**
45	5%	**0.95**	**42.75**
180	50%	**0.5**	90
220	30%	**0.7**	154

Solution 1.17

(a) (i) -9 (ii) 3 (iii) -2

(b) (i) 6 (ii) 2 (iii) 0

(c) (i) -12 (ii) 14 (iii) -27

(d) (i) -4 (ii) 5 (iii) -3

Solution 1.18

(a) $\frac{100^2}{25^2} = \left(\frac{100}{25}\right)^2 = 4^2 = 16$

(b) $\frac{4^4}{2^4} = \left(\frac{4}{2}\right)^4 = 2^4 = 16$

(c) $\frac{9^3}{3^3} = \left(\frac{9}{3}\right)^3 = 3^3 = 27$

Solution 1.19

(a) (i) $10^5 = 100\,000$ (ii) $10^2 = 100$
(iii) $(-10)^5 = -100\,000$ (iv) $10^1 = 10$
(v) $10^{-3} = 0.001$

(b) (i) $10^3 = 1000$ (ii) $10^{-3} = 0.001$
(iii) $(-10)^3 = -1000$ (iv) $10^1 = 10$
(v) $10^{-1} = 0.1$

Solution 1.20

Power form	Logarithmic form
$81 = 3^4$	$\mathbf{\log_3 81 = 4}$
$\mathbf{4 = 2^2}$	$\log_2 4 = 2$
$1 = 6^0$	$\mathbf{\log_6 1 = 0}$
$\mathbf{125 = 5^3}$	$\log_5 125 = 3$
$0.2 = 5^{-1}$	$\mathbf{\log_5 0.2 = -1}$
$5 = 5^1$	$\mathbf{\log_5 5 = 1}$
$\boldsymbol{8 = x^2}$	$\log_x 8 = 2$
$x = 2^{-4}$	$\boldsymbol{\log_2 x = -4}$

Solution 1.21

(a) (i) $16 = 2^4$, so $\log_2 16 = 4$.

(ii) $2 = 2^1$, so $\log_2 2 = 1$.

(iii) $0.5 = \frac{1}{2} = 2^{-1}$, so $\log_2 0.5 = -1$.

(b) (i) $1000 = 10^3$, so $\log_{10} 1000 = 3$.

(ii) $10 = 10^1$, so $\log_{10} 10 = 1$.

(iii) $0.1 = 10^{-1}$, so $\log_{10} 0.1 = -1$.

Solution 1.22

(a) (i) 100 000 (ii) 1 (iii) 100

(b) (i) 9 (ii) 3 (iii) $\frac{1}{3}$

Solution 1.23

(a) (i) 0.1900 (ii) 1.1794 (iii) 2.8658

(b) (i) 0.4375 (ii) 2.7157 (iii) 6.5989

Solution 1.24

(a) (i) 3.3628 (ii) 1.0053 (iii) 287.9387

(b) (i) 1.6933 (ii) 1.0023 (iii) 11.6966

Solution 1.25

(a) (i) $\log_{10} 36$ (ii) $\log_{10} 2$ (iii) $\log_{10} 125$

(b) (i) $\ln 105$ (ii) $\ln 2.5$ (iii) $\ln 81$

Solution 1.26

(a) (i) $\sqrt{100} = 10$

(ii) $\frac{9}{\sqrt{9}} = \sqrt{9} = 3$

(iii) $\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$

(b) (i) $\sqrt{200} = \sqrt{2 \times 100} = 10\sqrt{2}$

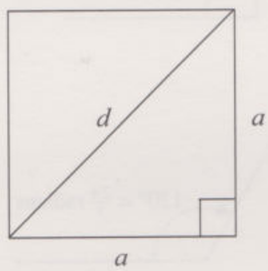
(ii) $\sqrt{112} = \sqrt{2 \times 2 \times 2 \times 2 \times 7} = 4\sqrt{7}$

(iii) $\sqrt{256} = \sqrt{4 \times 64}$
$= \sqrt{4 \times 4 \times 16}$
$= \sqrt{4} \times \sqrt{4} \times \sqrt{16} = 4 \times 4 = 16$

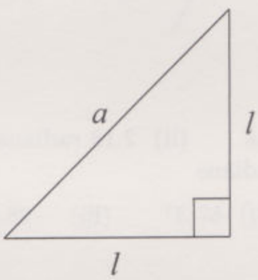
(iv) $\frac{\sqrt{15}}{\sqrt{3}} = \frac{\sqrt{5 \times 3}}{\sqrt{3}} = \sqrt{5}$

Solution 1.27

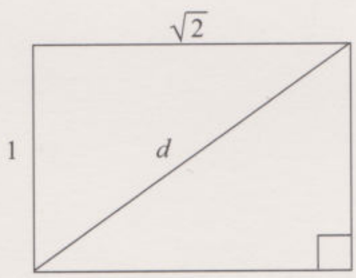
(a) If d is the length of the diagonal, then, by Pythagoras' Theorem, $2a^2 = d^2$. So the diagonal d has length $a\sqrt{2}$.



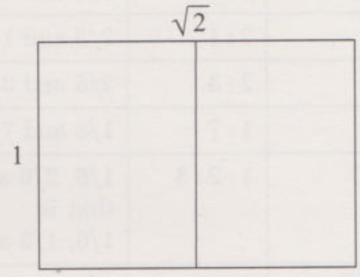
(b) If l is the length of the shorter edge, by Pythagoras' Theorem, $2l^2 = a^2$, so $l^2 = \frac{1}{2}a^2$. Hence the shorter edge l has length $a/\sqrt{2}$.



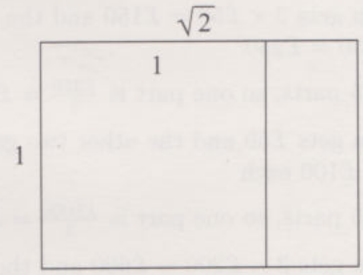
(c) Using Pythagoras' Theorem, $d^2 = 1^2 + (\sqrt{2})^2 = 1 + 2 = 3$. So the diagonal d has length $\sqrt{3}$.



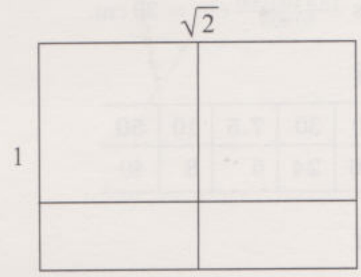
(d) The dimensions of each new rectangle are $\sqrt{2}/2 = 1/\sqrt{2}$ and 1. Thus the ratio of the edge lengths is $1 : 1/\sqrt{2} = \sqrt{2} : 1$. This is in the proportion of the original rectangle.



(e) The horizontal edge of the rectangle has length $\sqrt{2} - 1$ (and the vertical edge is 1).



(f) Each square has edge-length $\sqrt{2}/2 = 1/\sqrt{2}$, so each has area $(1/\sqrt{2}) \times (1/\sqrt{2}) = \frac{1}{2}$. The lower rectangle has length $\sqrt{2}$ and height $1 - 1/\sqrt{2} = (\sqrt{2} - 1)/\sqrt{2}$. Thus, the area of the rectangle is $\sqrt{2} \times (\sqrt{2} - 1)/\sqrt{2} = \sqrt{2} - 1$.



Check: The sum of these areas is $\frac{1}{2} + \frac{1}{2} + \sqrt{2} - 1 = \sqrt{2}$, which is the area of the initial rectangle.

Solution 1.28

(a) $\sqrt{2} \times \sqrt{24} = \sqrt{2} \times \sqrt{2} \times \sqrt{3} \times \sqrt{4} = 2 \times 2 \times \sqrt{3} = 4\sqrt{3}$

(b) $\sqrt{18}/3 = \sqrt{2} \times \sqrt{3} \times \sqrt{3}/3 = 3\sqrt{2}/3 = \sqrt{2}$

(c) $12/\sqrt{3} = 4 \times 3/\sqrt{3} = 4 \times \sqrt{3} \times \sqrt{3}/\sqrt{3} = 4\sqrt{3}$

Solution 1.29

Ratio (comparing part with part)	Ratio (in lowest terms)	As fractions (parts of the whole
$4:16$	$1:4$	1/5 and 4/5
$10:5$	$2:1$	2/3 and 1/3
$6:9$	$2:3$	2/5 and 3/5
$1:7$	$1:7$	1/8 and 7/8
$10:20:30$	$1:2:3$	1/6, 2/6 and 3/6; that is, 1/6, 1/3 and 1/2
$30:25$	$6:5$	6/11 and 5/11

Solution 1.30

(a) There are 8 parts, so one part is $\frac{£400}{8} = £50$.

One person gets $3 \times £50 = £150$ and the other gets $5 \times £50 = £250$.

(b) There are 5 parts, so one part is $\frac{£250}{5} = £50$.

One person gets £50 and the other two get $2 \times £50 = £100$ each

(c) There are 5 parts, so one part is $\frac{£1000}{5} = £200$.

One person gets $3 \times £200 = £600$ and the other gets $2 \times £200 = £400$.

Solution 1.31

(a) $$\begin{aligned} 16.5 \times 50\,000\,\text{cm} &= 16.5 \times 500\,\text{m} \\ &= 8\,250\,\text{m} \\ &= 8.25\,\text{km}. \end{aligned}$$

(b) $15\,\text{km} = 15 \times 100\,000\,\text{cm}$. This is represented on the map as $\frac{15 \times 100\,000}{50\,000}\,\text{cm} = 30\,\text{cm}$.

Solution 1.32

Height	15	20	30	**7.5**	10	**50**
Width	12	**16**	**24**	6	**8**	40

Solution 1.33

Since the ratio of Beth's contribution to Alan's stays at $2:1$, Alan's contribution will be half of Beth's – that is, $\frac{1}{2} \times £450 = £225$.

Solution 2.1

(a) (i) $16 > 2$: 16 is greater than 12.

(ii) $-11 < -9$: -11 is less than -9.

(b) (i) $-9 > -11$

(ii) $N \geq 20$, where N is the number of students in the classroom.

Solution 2.2

(a) $h = \dfrac{2A}{b}$. The diagrams are as follows.

$$h \xrightarrow{\times b} bh \xrightarrow{\div 2} \tfrac{bh}{2} = A$$
$$h = 2A/b \xleftarrow{\div b} 2A \xleftarrow{\times 2} A$$

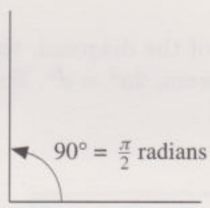
(b) (i) distance = speed × time

(ii) $\text{time} = \dfrac{\text{distance}}{\text{speed}}$

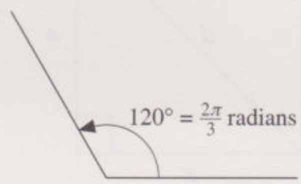
(c) $r = \dfrac{C}{2\pi}$

Solution 2.3

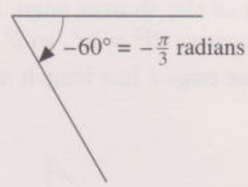
(a)



(b)



(c)



Solution 2.4

(a) (i) 0.94 radians (ii) 2.18 radians (iii) −11.64 radians

(b) (i) 51.4° (ii) 57.3° (iii) −28.6°

Solution 2.5

(a) $\frac{120}{5} = 24$ (b) $\frac{615}{6} = 102.5$

Solution 2.6

The mean number of siblings is $\frac{39}{33} \simeq 1.2$.

Solution 2.7

(a) Writing the numbers in ascending order gives the following.

21 23 24 26 26

The median is the middle number: 24.

(b) Writing the numbers in ascending order gives the following.

38 47 57 64 71 74 93

The median is the middle number: 64.

(c) Writing the numbers in ascending order gives the following.

4 6 7 7 8 9 10 12

The median is the mean of the two middle numbers, 7 and 8. Thus

$$\text{median} = \frac{7+8}{2} = 7.5.$$

(d) Writing the numbers in ascending order gives the following.

92 98 101 102 107 115

The median is the mean of the two middle numbers, 101 and 102. Thus

$$\text{median} = \frac{101+102}{2} = 101.5.$$

Solution 3.1

(a) $\theta = 180° - 100° - 30° = 50°$ (using ΔCDA);

$\phi = 180° - 40° - 30° = 110°$ (using ΔABC).

(b) $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 40° + 60° + 100° + 160° = 360°$.

Solution 3.2

In Figure 3.10, $\alpha = 101°$; scalene. In Figure 3.11, $\beta = 55°$ and $\gamma = 70°$; isosceles. In Figure 3.12, $\delta = 52°$; right-angled scalene.

Solution 3.3

(a) (i) By Pythagoras' Theorem, $BC^2 = AB^2 + AC^2 = 16 + 9 = 25$. Hence $BC = \sqrt{25} = 5\,\text{in}$.

(ii) By Pythagoras' Theorem, $YZ^2 = XY^2 + XZ^2$. So $XY^2 = YZ^2 - XZ^2 = 169 - 25 = 144$. Hence $XY = \sqrt{144} = 12\,\text{cm}$.

(iii) By Pythagoras' Theorem, $PR^2 = PQ^2 + QR^2$. So $PQ^2 = PR^2 - QR^2 = 4 - 1 = 3$. Hence $PQ = \sqrt{3} = 1.73\,\text{cm}$ (to 2 d.p.).

(b) If the square of the length of the longest side of a triangle is not equal to the sum of the squares of the lengths of the other two sides, then, by Pythagoras' Theorem, the triangle is not right-angled.

(i) $AB^2 + BC^2 = 4 + 9 = 13$, $AC^2 = 16$. Hence ΔABC is not right-angled.

(ii) $PQ^2 + PR^2 = (2.1)^2 + (3.5)^2 = 16.66$, $QR^2 = 16$. Hence ΔPQR is not right-angled.

Solution 3.4

(a) Perimeter of $ABCD = 2(4+3) = 14\,\text{cm}$

(b) Perimeter of $DBC = 3 + 4 + 5 = 12\,\text{cm}$

(c) Perimeter of $DOC = \frac{5}{2} + \frac{5}{2} + 4 = 9\,\text{cm}$

(d) Perimeter of $DON = \frac{3}{2} + 2 + \frac{5}{2} = 6\,\text{cm}$

Solution 3.5

(a) $C = 14\pi = 43.98\,\text{cm}$ (to 2 d.p.)

(b) $d = 44/\pi = 14.006\,\text{m}$ (to 3 d.p.)

Solution 3.6

(a) Area of $ABCD = 12\,\text{cm}^2$

(b) Area of $DBC = 6\,\text{cm}^2$

(c) Area of $BOC = 3\,\text{cm}^2$

(d) Area of $DON = 1.5\,\text{cm}^2$

Solution 3.7

(a) Area of circle $= \pi(30^2) = 900\pi = 2827.4\,\text{mm}^2$ (to 1 d.p.)

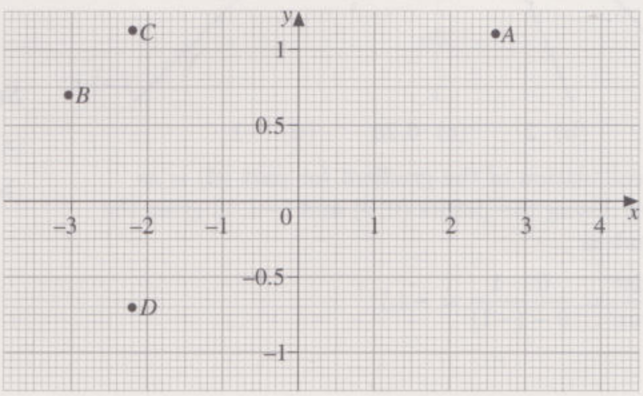
(b) $\frac{135}{360} = \frac{3}{8}$, so

$$\begin{aligned}\text{area of sector} &= \tfrac{3}{8}(900\pi)\\ &= 1060.3\,\text{mm}^2 \text{ (to 1 d.p.).}\end{aligned}$$

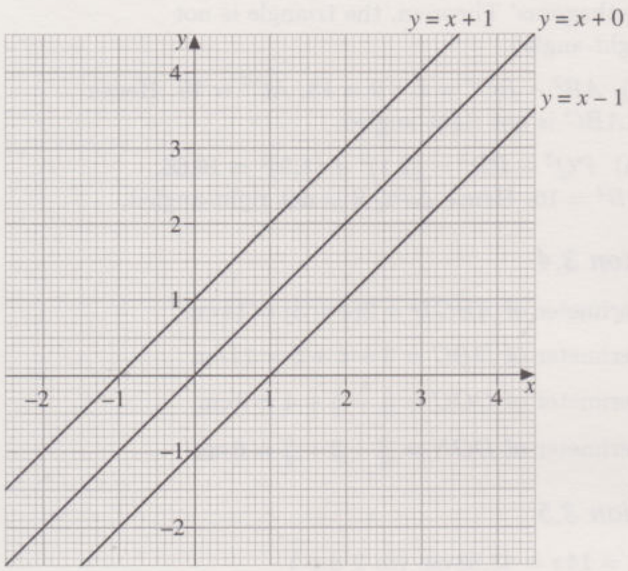
Solution 4.1

(a) $Q\,(-2.4,\ 0.7)$ $\quad S\,(3.6,\ -0.4)$

(b) The points A, B, C and D are shown below.



Solution 4.2



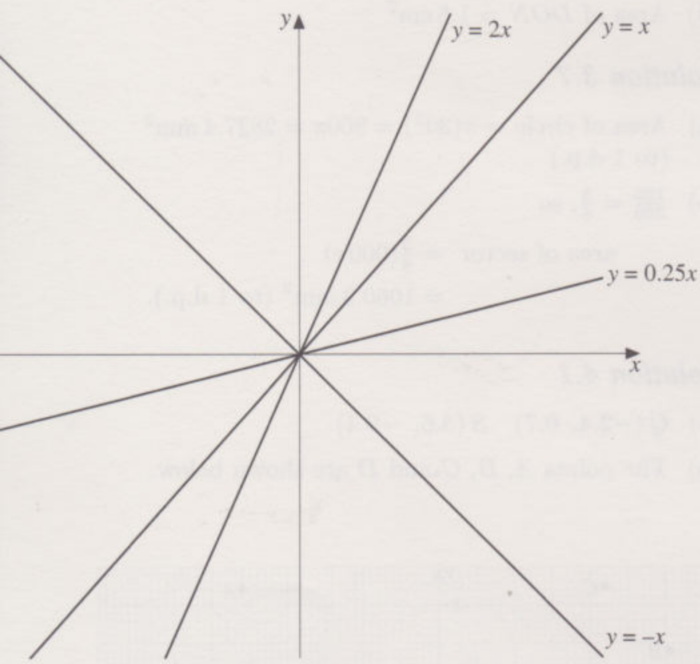
(a) gradient= 1, y-intercept 0.

(b) gradient= 1, y-intercept 1.

(c) gradient= 1, y-intercept -1.

(d) gradient= 1, y-intercept 0 (same line as $y = x$).

Solution 4.3



(a) 1 (b) 2 (c) 0.25 (d) -1

(The calculation of the gradient for part (d) is as follows. Using the points $(0, 0)$ and $(-1, 1)$ gives the gradient as

$$\frac{\text{rise}}{\text{run}} = \frac{0-1}{0-(-1)} = -1.)$$

Solution 5.1

(a) The variables are x and y. The coefficient of x is 4, the coefficient of y is 3, the constant is 5.

(b) The variables are k and n. The coefficient of k is 5, the coefficient of n is -3, the constant is -7.

(c) The only variable is a. The coefficient of a is $\frac{1}{2}$, the constant is $-\frac{2}{3}$.

Solution 5.2

(a)

Word expression	Algebraic expression
Think of a number	p
triple it	$3p$
add 10	$3p + 10$
double the result	$6p + 20$
subtract 8	$6p + 12$

(b)

Word expression	Algebraic expression
Think of a number	x (or any other letter)
square it	x^2
multiply by 4	$4x^2$
add twice the number you first thought of	$4x^2 + 2x$
add 8	$4x^2 + 2x + 8$

(c)

Word expression	Algebraic expression
Think of a number	x
subtract 2	$x - 2$
square the result	$(x-2)^2$
subtract 4 times the number you first thought of	$(x-2)^2 - 4x$
add 4	$(x-2)^2 - 4x + 4$

Solution 5.3

(a) (i) 68 (ii) 38 (iii) 47

(b) $4\frac{4}{15}$

Solution 5.4

(a) $21B$ (b) $10x$ (c) $6y$

(d) This expression cannot be simplified.

(e) $x^2 + 9x$ (It is usual to put terms in power order.)

(f) $6x^2 + 4x$ (g) $-\frac{1}{2}t$ (h) $4a + b^2$

Solution 5.5

(a) $5(2 + x) = 10 + 5x$

(b) $x(a + b) = xa + xb = ax + bx$

(c) $\frac{3}{4}(a - 4b) = \frac{3}{4}a - \frac{3}{4} \times 4b = \frac{3}{4}a - 3b$

(d) $a(a - 2) + a(a + 2) = a^2 - 2a + a^2 + 2a = 2a^2$

(e) $2x(y - x + 3) + 3xy = 2xy - 2x^2 + 6x + 3xy$
$= -2x^2 + 5xy + 6x$

(f) $a(b - a) + b(b - a) = ab - a^2 + b^2 - ba = -a^2 + b^2$
(ab is the same as ba.)

Do not worry if your solutions do not look quite the same; yours are right if the coefficients (including the signs) are correct even if you have the terms in a different order.

Solution 5.6

(a) $a^2 - 1$

(b) $y^2 + 8y + 7$

(c) $(p + 1)(p + 1) = p^2 + 2p + 1$

(d) $2x^2 + 11x + 14$

(e) $x^2 - 2x - 15$

(f) $10x^2 - 19x + 6$

(g) $6a^2 - a - 12$

(h) $6a^2 + a - 12$

(i) $(r - s)(r - s) = r^2 - 2rs + s^2$

(j) $(a + b)(a + b) = a^2 + 2ab + b^2$

(k) $a^2 - b^2$

(l) $a^2 - b^2$

Solution 5.7

(a) $4(a^2 - 3b)$

(b) $x(x + 1)$

(c) $4p(p + q)$

(d) $2\pi(r - 1)$

(e) $4xy(3 + z)$

(f) $\frac{\pi}{2}(3r - 1)$

(g) $x^2(8x - 1)$

(h) $2x^2(2x^2 + 1)$

Each of the next four expressions involves the difference of two squares.

(i) $(p + q)(p - q)$

(j) $(2x + y)(2x - y)$

(k) $(3s + 2t)(3s - 2t)$

(l) $\pi(r^2 - 9) = \pi(r + 3)(r - 3)$

Solution 5.8

(a) (i) $(x + 3)(x + 10)$

(ii) $(x - 4)(x - 1)$

(iii) $(x + 6)(x - 2)$

(iv) $(x - 6)(x + 2)$

(v) $(2x + 3)(2x - 3)$

(vi) $(x + 4)(3x - 1)$

(b) (i) $(x + 2)(x - 1)$

(ii) This expression does not factorise.

(iii) $x(x - 6)$

(iv) This expression does not factorise into factors involving whole numbers, but treating 12 as $\sqrt{12} \times \sqrt{12}$ it can be considered as the difference of two squares:
$x^2 - 12 = (x + \sqrt{12})(x - \sqrt{12})$.

(v) $(2x + 1)(x + 2)$

(vi) $(3x + 6)(x - 1) = 3(x + 2)(x - 1)$

Solution 5.9

(a) (i) $\dfrac{1}{z} + \dfrac{1}{x}$ (ii) $\dfrac{x^2}{2} - 3$ (iii) $2a - c$

(b) (i) $x^2 - 3$ (ii) $\dfrac{6x}{\sqrt{4x}} = \dfrac{6x}{\sqrt{4}\sqrt{x}} = 3\sqrt{x}$

(iii) $\dfrac{\sqrt{3x}}{\sqrt{3y}} = \sqrt{\dfrac{3x}{3y}} = \sqrt{\dfrac{x}{y}}$

Solution 5.10

(a) $\dfrac{1}{2} + \dfrac{1}{x} = \dfrac{x}{2x} + \dfrac{2}{2x} = \dfrac{x + 2}{2x}$

(b) $\dfrac{1}{x - 4} + \dfrac{1}{3x - 2}$

$$= \frac{3x - 2}{(x - 4)(3x - 2)} + \frac{x - 4}{(x - 4)(3x - 2)}$$

$$= \frac{3x - 2 + x - 4}{(x - 4)(3x - 2)} = \frac{4x - 6}{(x - 4)(3x - 2)}$$

(c) $\dfrac{3x+2}{x^2+1} - \dfrac{2x+1}{x+2}$

$$= \frac{(3x+2)(x+2)}{(x^2+1)(x+2)} - \frac{(x^2+1)(2x+1)}{(x^2+1)(x+2)}$$

$$= \frac{(3x^2+8x+4)-(2x^3+x^2+2x+1)}{(x^2+1)(x+2)}$$

$$= \frac{3x^2+8x+4-2x^3-x^2-2x-1}{(x^2+1)(x+2)}$$

$$= \frac{-2x^3+2x^2+6x+3}{(x^2+1)(x+2)}$$

Solution 5.11

(a) $x=2$ (b) $x=3$ (c) $x=\frac{3}{7}$

(d) $x=2$ (e) $x=3$ (f) $x=-4$

Solution 5.12

(a) $X=-5, Y=-12$ (b) $x=2, y=5$

(c) $x=2,\ y=1$ (d) $x=3, y=3$

(e) $p=-1, q=-2$

Solution 5.13

(a) $x=-2$ and $x=-3$ (b) $y=2$ and $y=8$

(c) $x=\frac{1}{2}$ and $x=-4$

(d) $p=-\frac{3}{2}$ (a single solution)

Solution 5.14

(a) The solutions are given by

$$x = \frac{-7 \pm \sqrt{7^2 - 4\times 2\times(-8)}}{4}$$

$$= \frac{-7\pm\sqrt{113}}{4}.$$

The solutions are $x=-4.408$ (to 3 d.p.) and $x=0.908$ (to 3 d.p.).

(b) This time there is only one solution $p=-\frac{3}{2}$.

(c) In this case 'b^2-4ac' is negative, so there are no real solutions.

Solution 5.15

The equation $\dfrac{1}{x-4}+\dfrac{1}{3x-2}=-\dfrac{3}{4}$ is equivalent to

$$\frac{1}{x-4}+\frac{1}{3x-2}+\frac{3}{4}=0.$$

The left-hand side is

$$\frac{1}{x-4}+\frac{1}{3x-2}+\frac{3}{4}$$

$$= \frac{4(3x-2)}{4(x-4)(3x-2)}+\frac{4(x-4)}{4(x-4)(3x-2)}+\frac{3(x-4)(3x-2)}{4(x-4)(3x-2)}$$

$$= \frac{12x-8+4x-16+3(3x^2-14x+8)}{4(x-4)(3x-2)}$$

$$= \frac{9x^2-26x}{4(x-4)(3x-2)}$$

$$= \frac{x(9x-26)}{4(x-4)(3x-2)}.$$

So $\dfrac{1}{x-4}+\dfrac{1}{3x-2}=-\dfrac{3}{4}$ is equivalent to

$$\frac{x(9x-26)}{4(x-4)(3x-2)}=0.$$

That is, $x(9x-26)=0$, so either $x=0$ or $x=26/9$.

Solution 5.16

Brief solutions are given.

(a)
$$\begin{aligned} 3^x &= 5 \\ x\log 3 &= \log 5 \\ x &= \frac{\log 5}{\log 3} \\ &= 1.465 \text{ (to 3 d.p.)} \end{aligned}$$

(b)
$$\begin{aligned} 5^x &= 3 \\ x &= \frac{\log 3}{\log 5} \\ &= 0.683 \text{ (to 3 d.p.)} \end{aligned}$$

(c)
$$\begin{aligned} 12^{-x} &= 5 \\ -x &= \frac{\log 5}{\log 12} \\ x &= \frac{-\log 5}{\log 12} \\ x &= -0.648 \text{ (to 3 d.p.)} \end{aligned}$$

(d)
$$\begin{aligned} 4^{x-1} &= 4 \\ (x-1)\log 4 &= \log 4 \\ x-1 &= 1 \\ x &= 2 \end{aligned}$$

Solution 5.17

(a) (i) x is greater than 180 and less than 360.

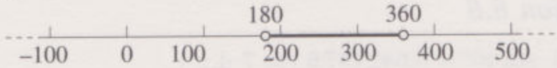
(ii) a is greater than or equal to 3.

(b) (i) Either $x > 0$, or $0 < x < \infty$.

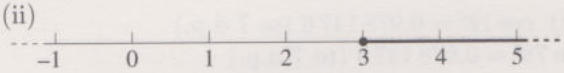
(ii) Either $x \geq -32$, or $-32 \leq x < \infty$.

(iii) $1 \leq a < 10$

(c) (i)



(ii)



Solution 5.18

(a) $x \geq 4$ (b) $x \leq \frac{1}{2}$ (c) $x > -\frac{8}{3}$

Solution 6.1

(a) The table is as follows.

Angle(°)	sin	cos	tan
0	0	1	0
30	0.5	0.8660	0.5774
45	0.7071	0.7071	1
60	0.8660	0.5	1.7321
90	1	0	
120	0.8660	−0.5	−1.7321
150	0.5	−0.8660	−0.5774
163	0.2924	−0.9563	−0.3057
180	0	−1	0
199	−0.3256	−0.9455	0.3443

Where the values in the above table are not exact, the values are given correct to four decimal places. The tangent ratio is not defined for $90°(\pi/2$ radians).

For the corresponding negative angles, the values for sine and tangent change sign, but the values for cosine remain the same.

(b) The table is as follows.

Angle (rad)	sin	cos	tan
0	0	1	0
0.25	0.2474	0.9689	0.2553
$\pi/6$	0.5	0.8660	0.5774
$\pi/4$	0.7071	0.7071	1
$4\pi/3$	−0.8660	−0.5	1.7321
1.5	0.9975	0.0707	14.1014
$\pi/2$	1	0	
2	0.9093	−0.4161	−2.1850
π	0	−1	0
2π	0	1	0
3π	0	−1	0

The remarks for part (a) apply here as well.

Solution 6.2

You should have noticed that each 'undoes' the other. So each is the inverse of the other. For example,

$$\sin \text{ is the inverse of } \sin^{-1}$$

and $\sin^{-1}$ is the inverse of $\sin$.

This 'undoing' property is the same for degrees and radians.

Solution 6.3

(a) (i) $\angle C = 90° - 58° = 32°$.

$\sin 58° = \dfrac{5.2}{BC}$, so

$$BC = \frac{5.2}{\sin 58°} = 6.1\,\text{cm (to 1 d.p.)},$$

$\tan 58° = \dfrac{5.2}{AB}$, so

$$AB = \frac{5.2}{\tan 58°} = 3.2\,\text{cm (to 1 d.p.)}.$$

(Trigonometric ratios of 32° could also be used to find these lengths.)

(ii) By Pythagoras' Theorem,

$$PQ^2 = 8^2 + 6^2 = 100.$$

So $PQ = \sqrt{100} = 10\,\text{cm}$.

$\sin P = \frac{6}{10} = 0.6$, so

$$P = \sin^{-1} 0.6 = 36.9° \text{ (to 1 d.p.)}.$$

(Other trigonometric ratios could be used.)

$\angle Q = 90° - 36.9° = 53.1°$ (to 1 d.p.).

(iii) $\angle X = 90° - 23° = 67°$.

$\cos 23° = \dfrac{YZ}{4.3}$, so

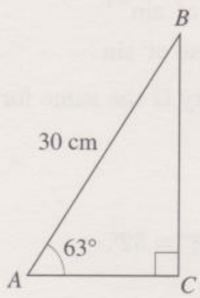
$$YZ = 4.3 \times \cos 23° = 4.0\,\text{cm (to 1 d.p.)}.$$

$\sin 23° = \dfrac{XY}{4.3}$, so

$$XY = 4.3 \times \sin 23° = 1.7\,\text{cm (to 1 d.p.)}.$$

(Other trigonometric ratios could be used.)

(b) Let the triangle be ABC with $\angle A = 63°$, as in the figure.



Then $\angle B = 90° - 63° = 27°$.

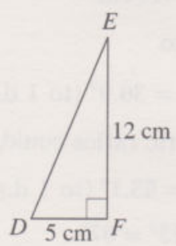
$\sin 63° = \dfrac{BC}{30}$, so

$$BC = 30 \times \sin 63° = 26.7\,\text{cm (to 1 d.p.)}.$$

$\cos 63° = \dfrac{AC}{30}$, so

$$AC = 30 \times \cos 63° = 13.6\,\text{cm (to 1 d.p.)}.$$

(c) Let the triangle be DEF with the right angle at F, as below.



Then $\tan D = \frac{12}{5} = 2.4$, so

$$\angle D = \tan^{-1} 2.4 = 1.18 \text{ radians (to 2 d.p.)}.$$

$\tan E = \frac{5}{12} = 0.41\dot{6}$, so

$$\angle E = \tan^{-1} 0.41\dot{6} = 0.39 \text{ radians (to 2 d.p.)}.$$

Solution 6.4

(a) (i) $\cot 40° = \dfrac{1}{\tan 40°} = 1.1918$ (to 4 d.p.)

(ii) $\sec 20° = \dfrac{1}{\cos 20°} = 1.0642$ (to 4 d.p.)

(iii) $\operatorname{cosec} 82° = \dfrac{1}{\sin 82°} = 1.0098$ (to 4 d.p.)

(b) (i) $\operatorname{cosec} 1.2 = 1.0729$ (to 4 d.p.)

(ii) $\cot\left(\frac{\pi}{5}\right) = 1.3764$ (to 4 d.p.)

(iii) $\sec\left(\frac{\pi}{8}\right) = 1.0824$ (to 4 d.p.)

Solution 6.5

(a) $XZ = \sqrt{2}$, and

$$\angle Z = \angle X = \frac{\pi}{4} \text{ radians.}$$

(b) $\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$, $\tan\dfrac{\pi}{4} = 1$,

$\operatorname{cosec}\dfrac{\pi}{4} = \sqrt{2}$, $\sec\dfrac{\pi}{4} = \sqrt{2}$, $\cot\dfrac{\pi}{4} = 1$.

Solution 6.6

(a) (i) $\sin 40° = 0.642\,787\,6$ (to 7 d.p.)
$\cos 50° = 0.642\,787\,6$ (to 7 d.p.)

(ii) $\cos 12° = 0.978\,147\,6$ (to 7 d.p.)
$\sin 78° = 0.978\,147\,6$ (to 7 d.p.)

(b) Since $\angle A + \angle B = 90°$, these occur in a right-angled triangle ABC, as shown below.

Then, in the usual notation, $\sin A = a/c$ and $\cos B = a/c$.

So $\sin A = \cos B$.

Also $\cos A = b/c$ and $\sin B = b/c$.

So $\cos A = \sin B$.

Solution 6.7

$\sin\theta = a/c$, so $\sin^2\theta = a^2/c^2$.

$\cos\theta = b/c$, so $\cos^2\theta = b^2/c^2$.

Hence

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \frac{a^2}{c^2} + \frac{b^2}{c^2}\\ &= \frac{a^2+b^2}{c^2}\\ &= \frac{c^2}{c^2} \quad (a^2+b^2=c^2,\\ &\qquad\text{by Pythagoras' Theorem})\\ &= 1.\end{aligned}$$

Solution 6.8

The area of triangle ABC is

$$\tfrac{1}{2} \times 10.3 \times 6.1 \times \sin 50° = 24.1\,\text{cm}^2 \text{ (to 1 d.p.)}.$$

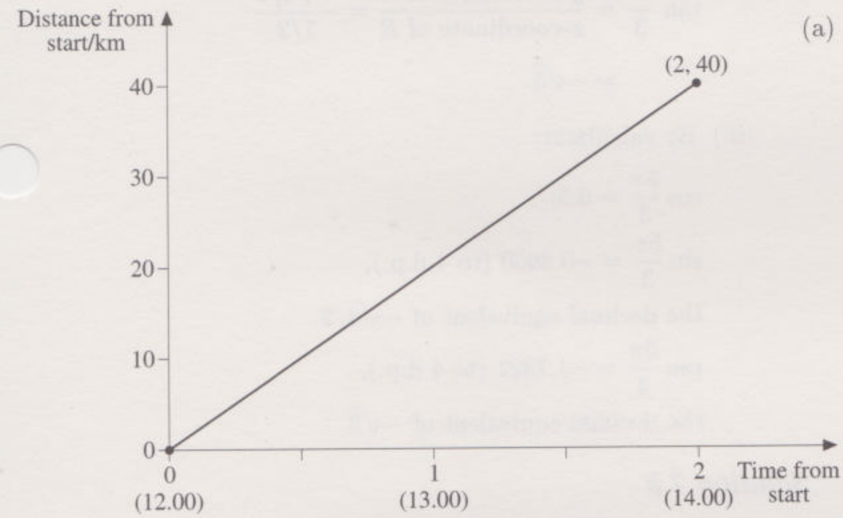
Solution 7.1

The gradient of the graph for the stage from C to D is zero, so the speed for this stage is zero. C and D are the same physical point of the journey; this point is the furthest point from the starting point.

She returns from D to her starting point at an average speed, indicated by the slope from D to E. (The slope is downwards, since she is now moving in the opposite direction to that chosen as positive.)

Note that the points A and E are the same physical points of the journey.

Solution 7.2



The distance–time graph is shown above. The straight line starts at the origin of the graph, representing the ferry's departure from the English port at 12.00 and finishes at the point $(2, 40)$ representing its arrival at the French port at 14.00.

(a) The ferry has gone 15 km by 12.45, and 35 km by 13.45.

(b) By 12.15 the ferry is 5 km away from the English port, and by 13.10 it is about 23 km away.

(c) The gradient of the graph is

$$\frac{40}{2} = 20\,\text{km per hour.}$$

(d) The ferry's average speed is 20 km per hour.

Solution 7.3

(a) $A = \pi r^2$, that is $A \propto r^2$ or $A = kr^2$, where $k = \pi$. The graph will be of the form of Figure 7.5.

(b) $r = \sqrt{A/\pi}$, that is, $r \propto \sqrt{A}$ or $r = kA^{\frac{1}{2}}$, where $k = \sqrt{1/\pi}$. The graph will be of the form of Figure 7.6.

(c) $s \times t = 650$, so $s \propto 1/t$, or $s = k/t = kt^{-1}$, where $k = 650$. The graph will be of the form of Figure 7.7.

(d) $D = \dfrac{1}{0.66}P$, that is $D \propto P$ or $D = kP$, where $k = 1/0.66$. The graph will be of the form of Figure 7.4.

Solution 7.4

(a) $f(-4) = 49$, $f(-1) = 4$, $f(0) = 1$, $f(1) = 4$, $f(4) = 49$.

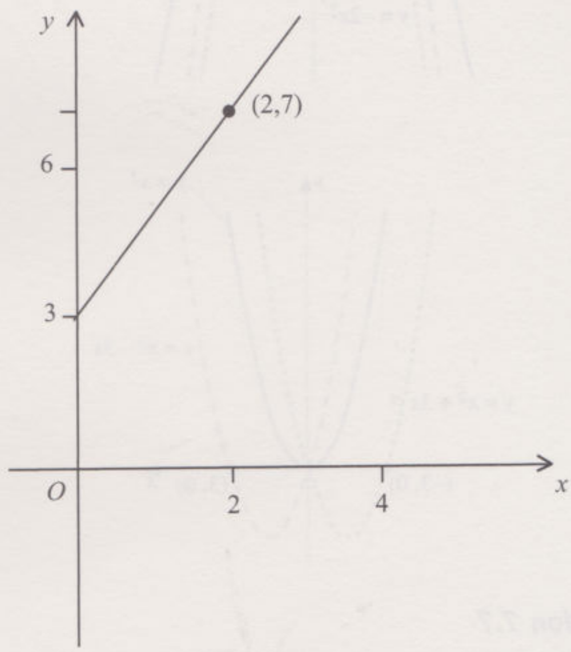
Note that for each value of x, there is just one value of $f(x)$.

(b) $g(0) = 0$, $g(1) = 1$, $g(4) = 2$.

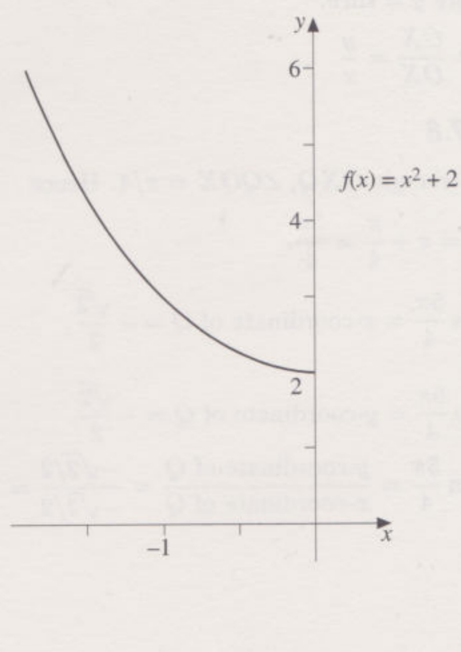
The numbers -4 and -1 are not in the domain of the function $g(x) = \sqrt{x}$.

Solution 7.5

(a)

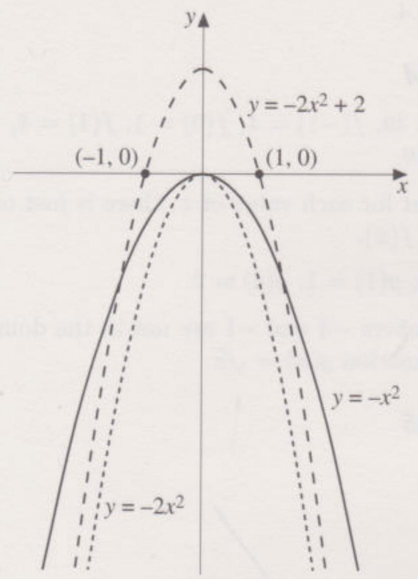


(b)

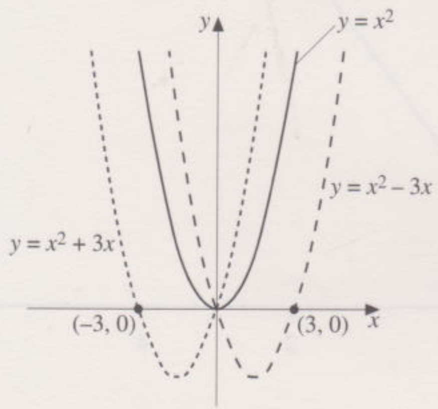


Solution 7.6

(a)



(b)



Solution 7.7

(a) We have $y = CX$ and $\sin\theta = \dfrac{CX}{1}$; therefore $y = \sin\theta$.

(b) $\tan\theta = \dfrac{CX}{OX} = \dfrac{y}{x}$

Solution 7.8

(a) (i) In triangle OXQ, $\angle QOX = \pi/4$. Hence

$$\alpha = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

(ii) $\cos\dfrac{5\pi}{4} = x\text{-coordinate of } Q = -\dfrac{\sqrt{2}}{2}.$

$$\sin\frac{5\pi}{4} = y\text{-coordinate of } Q = -\frac{\sqrt{2}}{2}.$$

$$\tan\frac{5\pi}{4} = \frac{y\text{-coordinate of } Q}{x\text{-coordinate of } Q} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1.$$

(iii) By calculator,

$$\cos\frac{5\pi}{4} = \sin\frac{5\pi}{4} = -0.7071 \text{ (to 4 d.p.)},$$

which is the decimal equivalent of $-\sqrt{2}/2$.

$$\tan\frac{5\pi}{4} = 1.$$

(b) (i) In triangle OXR, $\angle ROX = \pi/3$. Hence

$$\beta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

(ii) $\cos\dfrac{5\pi}{3} = x\text{-coordinate of } R = \dfrac{1}{2}.$

$$\sin\frac{5\pi}{3} = y\text{-coordinate of } R = -\frac{\sqrt{3}}{2}.$$

$$\tan\frac{5\pi}{3} = \frac{y\text{-coordinate of } R}{x\text{-coordinate of } R} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}.$$

(iii) By calculator,

$$\cos\frac{5\pi}{3} = 0.5;$$

$$\sin\frac{5\pi}{3} = -0.8660 \text{ (to 4 d.p.)},$$

the decimal equivalent of $-\sqrt{3}/2$,

$$\tan\frac{5\pi}{3} = -1.7321 \text{ (to 4 d.p.)},$$

the decimal equivalent of $-\sqrt{3}$.

Solution 7.9

(a) (i) $e^1 = 2.718\,28\ldots$ and

$\ln(2.718\,28\ldots) = 1$

(ii) $e^2 = 7.389\,05\ldots$ and

$\ln(7.389\,05\ldots) = 2$

(b) (i) $\ln 1 = 0$ and

$e^0 = 1$

(ii) $\ln 2 = 0.693\,14\ldots$ and

$e^{0.693\,14\ldots} = 2.$

(c) In each case, the starting number is the final number.

Solution 8.1

The angle sum of the plane figures is as follows.

(a) A heptagon: $(7-2)\times 180° = 900°$

(b) A nonagon: $(9-2)\times 180° = 1260°$

(c) A decagon: $(10-2)\times 180° = 1440°$

Solution 8.2

(a) $\triangle PXY$ is similar to $\triangle XYZ$ and $\triangle PXZ$.

(b) We have

$$\sin \angle XZY = \frac{XY}{YZ} \text{ and } \sin \angle PZX = \frac{PX}{XZ}.$$

Since triangles XYZ and PXZ are similar,

$$\begin{aligned}\frac{XY}{YZ} &= \frac{kPX}{kXZ}, \text{ where } k \text{ is constant,}\\ &= \frac{PX}{XZ}.\end{aligned}$$

Thus $\sin \angle XZY = \sin \angle PZX$.

Solution 8.3

(a) $S = 2(6 \times 7) + 2(6 \times 10) + 2(7 \times 10)$
$= 344\,\text{cm}^2$

(b) $S = 2(\pi \times 10^2) + 2\pi \times 10 \times 40$
$= 3141.59\,\text{cm}^2$ (to 2 d.p.)

(c) $S = 4\pi \times 3^2 = 36\pi$
$= 113.10\,\text{cm}^2$ (to 2 d.p.)

Solution 8.4

(a) $V = (6 \times 7) \times 10 = 420\,\text{cm}^3$.

(b) $V = (\pi \times 10^2) \times 40$
$= 12566.37\,\text{cm}^3$ (to 2 d.p.)

(c) $V = \frac{4}{3}\pi \times 4^3 = \frac{256}{3}\pi$
$= 268.08\,\text{cm}^3$ (to 2 d.p.)

Index

REVISION PACK BOOKMARK

Take charge of your own learning

Organise

your time – plan what, when, where and for how long;
study materials, notes, study records, other resources.

Know what to do when stuck

sort out what you do know;
try to break down the problem into manageable parts;
use one or more of these strategies –
explain to a friend, talk to a tutor, take a break,
skip over the problem (later studies may help).

Beware of

filling in time rather than constructively studying;
telling yourself you understand when you don't;
letting schedules slip.

Be reflective

review progress regularly;
make revision notes as you go;
review planning and studying organisation.

Look after your key resources – motivation, energy, concentration